

# Milnor operations and the generalized Chern character

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We have shown that the  $n$ -th Morava  $K$ -theory  $K^*(X)$  for a CW-spectrum  $X$  with action of Morava stabilizer group  $G_n$  can be recovered from the system of some height- $(n + 1)$  cohomology groups  $E^*(Z)$  with  $G_{n+1}$ -action indexed by finite subspectra  $Z$ . In this note we reformulate and extend the above result. We construct a symmetric monoidal functor  $\mathcal{F}$  from the category of  $E_*^\vee(E)$ -precomodules to the category of  $K_*(K)$ -comodules. Then we show that  $K^*(X)$  is naturally isomorphic to the inverse limit of  $\mathcal{F}(E^*(Z))$  as a  $K_*(K)$ -comodule.

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*Dedicated to Professor Nishida on the occasion of his 60th birthday*

## 1 Introduction

From the chromatic point of view, the complex  $K$ -theory is a height-1 cohomology theory and the ordinary rational cohomology is a height-0 cohomology theory. As geometric aspects, the rational cohomology is defined by means of differential forms, and the  $K$ -theory is defined by means of vector bundles. The classical Chern character associates to a complex vector bundle the sum of exponentials of formal roots of the total Chern polynomial. It may be regarded as a multiplicative natural transformation from the  $K$ -theory to the rational cohomology, that is to say, from a height-1 cohomology to a height-0 cohomology. There is a height-2 cohomology theory, which is called the elliptic cohomology. Conjecturally, the elliptic cohomology may also have a geometric interpretation analogous to the rational cohomology and the  $K$ -theory. A generalization of Chern character to the elliptic cohomology has been considered by Miller [15]. The idea is that the formal group law on the moduli stack of elliptic curves is degenerate to the multiplicative formal group law when it is restricted around a cusp. Miller's elliptic character is a multiplicative natural transformation from the elliptic cohomology to the  $K$ -theory with coefficients in the formal Laurent series ring, hence from a height-2 theory to a height-1 theory.

The elliptic character may be regarded as the  $q$ -expansion map of modular forms parametrized by spaces. The  $q$ -expansion is the Fourier expansion of modular forms

at a cusp, which associates a formal Laurent series with variable  $q = \exp(2\pi\sqrt{-1}\tau)$ . The  $q$ -expansion map has been extended at more general algebraic setting, and it has been shown that it has a very good property, which is called  $q$ -expansion principle (cf Deligne–Rapoport [3] and Katz [11]). In particular, the  $q$ -expansion map is injective, and hence the modular forms are controlled by their  $q$ -expansion. The analogous property on the elliptic character has been studied at odd primes by Laures [13]. At the prime 2, there is a more elaborate cohomology theory related to elliptic curves and modular forms. It is defined by the spectrum  $tmf$  of topological modular forms, which is introduced by Mike Hopkins. The  $q$ -expansion map (evaluation at the Tate curve) also induces a ring spectrum map from  $tmf$  to  $K[[q]]$ . In [14] Laures studied the  $K(1)$ -local topological modular forms at the prime 2, and discussed the relationship between the  $q$ -expansion map, Witten genus and  $MO\langle 8 \rangle$ -orientation of  $tmf$ .

A generalization of Chern character to higher chromatic level has been considered by Ando, Morava and Sadofsky [1] under geometric background. Their generalized Chern character is a multiplicative natural transformation from  $(n+1)$ -th Morava  $E$ -theory  $E_{n+1}$  to the  $n$ -th Morava  $E$ -theory with coefficients in some big Cohen ring. In [23] we studied the degeneration of formal group law, which is used to construct their Chern character. By using the results in [23] we refined their generalized Chern character in [22]. Then we were able to control it algebraically. In this note we reformulate and extend some results in [22].

Let  $\mathcal{S}$  be the stable homotopy category of  $p$ -local spectra for some prime  $p$ . It is known that there is a filtration of full subcategories of  $\mathcal{S}$ , which corresponds to the height filtration of the moduli space of one-dimensional commutative formal group laws; see Devinatz, Hopkins and Smith [5], Hopkins and Smith [7], Morava [17] and Ravenel [19]. The layers of this filtration are equivalent to the  $K(n)$ -local categories, where  $K(n)$  is the  $n$ -th Morava  $K$ -theory. Hence it is considered that the stable homotopy category  $\mathcal{S}$  is built from  $K(n)$ -local categories. In fact, the chromatic convergence theorem (cf Ravenel [19]) says that for a  $p$ -local finite spectrum  $X$ , the natural tower  $\cdots \rightarrow L_{n+1}X \rightarrow L_nX \rightarrow \cdots \rightarrow L_0X$  recovers  $X$ , that is, the homotopy inverse limit of the tower is homotopy equivalent to  $X$ . Furthermore, the chromatic splitting conjecture (cf Hovey [8]) implies that the  $p$ -completion of a finite spectrum  $X$  is a direct summand of the product  $\prod_n L_{K(n)}X$ . This means that it is not necessarily to reconstruct the tower but it is sufficient to know all  $L_{K(n)}X$  to obtain some information about  $X$ . In [8] Hovey observed that the weak form of the chromatic splitting conjecture should imply many interesting results. The weak form means that the canonical map  $L_n(S^0)_p^\wedge \rightarrow L_n L_{K(n+1)}S^0$  is a split monomorphism. In [16, Remark 3.1(i)] Minami indicated that the weak form implies that there is a natural map  $\rho_X$  for a finite spectrum  $X$  from the  $K(n+1)$ -localization  $L_{K(n+1)}X$  to the  $K(n)$ -localization  $L_{K(n)}X$  such that

the following diagram commutes:

$$(1) \quad \begin{array}{ccc} & X & \\ \eta_{K(n+1)} \swarrow & & \searrow \eta_{K(n)} \\ L_{K(n+1)}X & \xrightarrow{\rho} & L_{K(n)}X \end{array}$$

where  $\eta_{K(n)}: X \rightarrow L_{K(n)}X$  and  $\eta_{K(n+1)}: X \rightarrow L_{K(n+1)}X$  are the localization maps.

In [22] we considered the modulo  $I_n$  version of the algebraic analogue of the diagram (1). The Morava  $E$ -theory  $E_n$  defines a functor from the  $K(n)$ -local category to the category of twisted  $E_{n*}-G_n$ -modules, where  $G_n$  is the  $n$ -th extended Morava stabilizer group. The Adams–Novikov spectral sequence based on  $E_n$ -theory has its  $E_2$ -term  $H_c^{**}(G_n; E_{n*}(X))$  and converges to  $\pi_*(L_{K(n)}X)$  strongly if  $X$  is finite, where  $H_c^{**}(G_n; -)$  is the continuous cohomology group of  $G_n$ . Hence the category of twisted  $E_{n*}-G_n$ -modules can be considered as an algebraic approximation of the  $K(n)$ -local category.

Let  $BP_*$  be the Brown–Peterson spectrum at an odd prime  $p$  and  $I_n$  the invariant prime ideal generated by  $p, v_1, \dots, v_{n-1}$ . There is a commutative ring spectrum  $E = E_{n+1}/I_n$ , which is a complex oriented cohomology theory with coefficient ring  $E_* = E_{n+1,*}/I_n = \mathbf{F}[[u_n]][u^{\pm 1}]$ . We denote by  $K^*(-)$  a variant of  $n$ -th Morava  $K$ -theory with coefficient  $\mathbf{F}[w^{\pm 1}]$ . The modulo  $I_n$ -version of the algebraic analogue of the diagram (1) is

$$(2) \quad \begin{array}{ccc} & \mathcal{S}^f & \\ E \swarrow & & \searrow K \\ \mathcal{M}_{n+1}^f & \xrightarrow{\mu} & \mathcal{M}_n^f \end{array}$$

where  $\mathcal{S}^f$  is the stable homotopy category of finite spectra,  $\mathcal{M}_{n+1}^f$  is the category of finitely generated twisted  $E_*-G_{n+1}$ -modules and  $\mathcal{M}_n$  is the category of finitely generated twisted  $K_*-G_n$ -modules. So the question is: Does there exist a functor  $\mu$  from  $\mathcal{M}_{n+1}^f$  to  $\mathcal{M}_n^f$  which makes the diagram (2) commute?

In [23] we constructed a Galois extension  $L$  of the quotient field of  $\mathbf{F}[[u_n]]$ , over which the formal group  $F_{n+1}$  associated with  $E$  is nicely isomorphic to the Honda group law  $H_n$ . By using this result, we constructed a natural transformation  $\Theta$  from  $E$ -cohomology to  $K$ -cohomology with coefficients in  $L$  in [22]:

$$\Theta: E^*(X) \longrightarrow L^*(X).$$

This is regarded as a generalized Chern character since it is a multiplicative natural transformation from the height- $(n+1)$  cohomology  $E$  to the height- $n$  cohomology

$K$  with coefficients in  $L$ . Then it is shown that  $\Theta$  is equivariant with respect to the action of  $G_{n+1}$ , and  $\Theta$  induces  $L_* \otimes_{E_*} E^*(Z) \cong L_* \otimes_{K_*} K^*(Z)$ , a natural isomorphism of  $\mathcal{G}$ -modules for any finite spectrum  $Z$ , where  $\mathcal{G} = \Gamma \ltimes (S_n \times S_{n+1})$  and  $L_* = L[u^{\pm 1}]$ . By these results, we have shown that there is a natural isomorphism of twisted  $K_*-G_n$ -modules:

$$K^*(X) \cong \varprojlim_Z H^0(S_{n+1}; L_* \otimes_{E_*} E^*(Z))$$

for any CW-spectrum  $X$  where  $Z$  ranges over finite subspectra of  $X$  [22, Corollary 4.3]. Hence if we set the functor  $\mu(-) = H^0(S_{n+1}; L_* \otimes_{E_*} -)$ , it makes the diagram (2) commute.

Essentially, the twisted  $K_*-G_n$ -module structure gives  $K^*(X)$  the stable cohomology operations except for Milnor operations. So in this note we would like to extend the above result in the form which includes the action of Milnor operations. Note that the twisted  $K_*-G_n$ -module structure on  $K^*(X)$  with Milnor operations is equivalent to the  $K_*(K)$ -comodule structure. In this note we construct a symmetric monoidal functor  $\mathcal{F}$  from the category of profinite  $E_*^\vee(E)$ -precomodules to the category of profinite  $K_*(K)$ -comodules. Roughly speaking, a profinite  $E_*^\vee(E)$ -precomodule is a filtered inverse limit of finitely generated  $E_*$ -module

$$M = \varprojlim M/M_\lambda$$

such that  $M^c$  has a complete  $E_*^\vee(E)$ -comodule structure, where

$$M^c = \varprojlim M/M_\lambda + \mathfrak{m}^i M.$$

For a profinite  $E_*^\vee(E)$ -precomodule  $M$ , there is a natural twisted  $E_*-G_{n+1}$ -module structure on  $M$ . Hence  $M \widehat{\otimes}_{E_*} L_*$  is a twisted  $L_*-G_{n+1}$ -module. We set

$$\mathcal{F}(M) = H^0(S_{n+1}; M \widehat{\otimes}_{E_*} L_*).$$

**Theorem A (Corollary 6.5)** *The functor  $\mathcal{F}$  extends to a symmetric monoidal functor from the category of profinite  $E_*^\vee(E)$ -precomodules to the category of profinite  $K_*(K)$ -comodules.*

For a spectrum  $X$  we denote by  $\Lambda(X)$  the category whose objects are maps  $Z \xrightarrow{u} X$  with  $Z$  finite. We associate to  $X$  a cofiltered system  $\mathbb{E}^*(X) = \{E^*(Z)\}$  and  $\mathbb{K}^*(X) = \{K^*(Z)\}$  indexed by  $\Lambda(X)$ . Then  $E^*(X) = \varprojlim \mathbb{E}^*(X)$  and  $K^*(X) = \varprojlim \mathbb{K}^*(X)$ . The following is the main theorem of this note.

**Theorem B** (Theorem 6.11) *For any spectrum  $X$ , the generalized Chern character  $\Theta$  induces a natural isomorphism of cofiltered system of  $K_*(K)$ –comodules:*

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

*If  $X$  is a space, then this is an isomorphism of systems of  $K_*(K)$ –comodule algebras.*

The organization of this note is as follows. In Section 2.1 we summarize well-known results on generalized cohomology theories which are Landweber exact over  $P(n)$ . In Section 2.2 we review our main result in [23] on degeneration of formal group laws. In Section 2.3 we review on the construction of the generalized Chern character.

In Section 3.1 we study the category of complete Hausdorff filtered modules and the action of a profinite group on a complete module. In Section 3.2 we recall complete Hopf algebroids and their comodules.

In Section 4.1 we describe the structure of complete Hopf algebroid  $C(G, R^c)$ , where  $C(G, R^c)$  is the ring of all continuous functions from a profinite group  $G$  to an even-periodic complete local ring  $R^c$ . In Section 4.2 we show the well-known fact that the category of complete  $C(G, R^c)$ –comodules is equivalent to the category of complete twisted  $R^c$ – $G$ –modules. Usually, we use and study the category of complete twisted  $\mathbf{F}_{p^n}$ – $G_n$ –modules. In Section 4.3 we show that there is no essential difference between the category of complete twisted  $\mathbf{F}$ – $G_n$ –modules and the category of complete twisted  $\mathbf{F}_{p^n}$ – $G_n$ –modules. In Section 4.4 we reformulate a result of [22]. We construct a symmetric monoidal functor  $\mathcal{F}$  from the category of profinite  $C(G_{n+1}, E_*^c)$ –precomodules to the category of profinite  $C(G_n, K_*)$ –comodules, and show that there is a natural isomorphism between  $\mathcal{F}(\mathbb{E}^*(X))$  and  $\mathbb{K}^*(X)$  as systems of  $C(G_n, K_*)$ –comodules.

In Section 5.1 we define a complete co-operation ring  $A_*^\vee(A)$  for  $A = E_{n+k}/I_n$ , and study a  $A_*^\vee(A)$ –(pre)comodule algebra structure on the  $A$ –cohomology of the projective space  $\mathbf{C}P^\infty$  and the lens space  $S^{2p^n-1}/(\mathbf{Z}/p)$ . In Section 5.2 we study a twisted  $E_*$ – $G_{n+1}$ –module structure on the exterior algebra  $\Lambda_{E_*}$  and show that  $\mathcal{F}(\Lambda_{E_*}) \cong \Lambda_{K_*}$  as twisted  $K_*$ – $G_n$ –modules. In Section 5.3 we define Milnor operations  $Q_i^A$  for a  $\Lambda_{A_*}$ –comodule  $M$ . In Section 5.4 we study  $A_*^\vee(A)$ –comodule structures in terms of  $C_{A_*}$ –comodule structures and  $\Lambda_{A_*}$ –comodule structures. We show that an  $A_*^\vee(A)$ –comodule structure is equivalent to a  $C_{A_*}$ –comodule structure and a  $\Lambda_{A_*}$ –comodule structure which satisfy some compatibility condition.

In Section 6.1 we extend the symmetric monoidal functor  $\mathcal{F}$  from the category of profinite  $E_*^\vee(E)$ –precomodules to the category of profinite  $K_*(K)$ –comodules. In Section 6.2 we prove the main theorem.

In this note  $p$  shall be an odd prime except for [Section 2.2](#),  $\mathbf{F}$  a finite field containing  $\mathbf{F}_{p^n}$  and  $\mathbf{F}_{p^{n+1}}$ , and  $\text{Gal}$  the Galois group  $\text{Gal}(\mathbf{F}/\mathbf{F}_p)$ . We think a group  $G$  acts on a ring  $R$  from the right and denote by  $r^g$  the right action of  $g \in G$  on  $r \in R$ . For a power series  $\alpha(X) = \sum \alpha_i X^i \in R[[X]]$ , we set  $\alpha^g(X) = \sum \alpha_i^g X^i$  for  $g \in G$ . An  $R$ -module means a left  $R$ -module if nothing else is indicated.

## 2 The generalized Chern character

### 2.1 Landweber exact theories over $P(n)$

**Definition 2.1** A graded commutative ring  $R_*$  is said to be even-periodic if  $R_*$  is concentrated in even degrees and  $R_2$  contains a unit in  $R_*$ . A multiplicative generalized cohomology theory  $h^*(-)$  is said to be even-periodic if the coefficient ring  $h_* = h^*(\text{pt})$  is even-periodic.

For a spectrum  $X$ , we denote by  $\Lambda(X)$  the category whose objects are maps  $Z \xrightarrow{u} X$  such that  $Z$  is finite, and whose morphisms are maps  $Z \xrightarrow{v} Z'$  such that  $u'v = u$ . Then  $\Lambda(X)$  is an essentially small filtered category.

**Definition 2.2** Let  $h^*(-)$  be a generalized cohomology theory. For a spectrum  $X$ , we define a filtration on  $h^*(X)$  indexed by  $\Lambda(X)$  as

$$F^Z h^*(X) = \text{Ker}(h^*(X) \longrightarrow h^*(Z))$$

for  $Z \in \Lambda(X)$ . We call this filtration the profinite filtration and the resulting topology the profinite topology.

**Remark 2.3** If  $h^*(-)$  is even-periodic and the degree-0 coefficient ring  $h_0$  is a complete Noetherian local ring, then  $h^*(Z)$  is a finitely generated  $h_*$ -module for all  $Z \in \Lambda(X)$ , and the canonical homomorphism

$$h^*(X) \longrightarrow \varprojlim_{Z \in \Lambda(X)} h^*(Z)$$

is an isomorphism. This implies that  $h^*(X)$  is complete Hausdorff with respect to the profinite topology.

Let  $BP$  be the Brown-Peterson spectrum at an odd prime  $p$ , whose coefficient ring is given by  $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  with  $|v_i| = 2(p^i - 1)$ . Let  $I_n$  be the invariant prime ideal generated by  $p, v_1, \dots, v_{n-1}$ . There is a commutative  $BP$ -algebra spectrum  $P(n)$ ,

whose coefficient ring is  $P(n)_* = BP_*/I_n$ . In particular,  $P(0) = BP$ . Let  $\mathcal{X} = \mathbf{C}P^\infty$  the infinite dimensional complex projective space, and  $\mathcal{Y} = S^{2p^n-1}/(\mathbf{Z}/p)$  the lens space of dimension  $2p^n - 1$ , where  $\mathbf{Z}/p$  is the cyclic group of order  $p$  acting on the unit sphere  $S^{2p^n-1}$  in  $\mathbf{C}^{p^n}$  by standard way. These spaces are important test spaces to stable cohomology operations of complex oriented cohomology theories (cf [2, Section 14]). The  $P(n)$ -cohomology of  $\mathcal{X}$  and  $\mathcal{Y}$  are given as follows:

$$\begin{aligned} P(n)^*(\mathcal{X}) &= P(n)_* \llbracket x \rrbracket, \\ P(n)^*(\mathcal{Y}) &= \Lambda(y) \otimes P(n)_*[x]/(x^{p^n}), \end{aligned}$$

where  $x \in P(n)^2(\mathcal{X})$  is the orientation class and  $y \in P(n)^1(\mathcal{Y})$ .

Let  $F$  be a  $p$ -typical formal group law over a commutative ring  $R$ . By universality of the  $p$ -typical formal group law  $F_{BP}$  associated to  $BP$ , there is a unique ring homomorphism  $f: BP_* \rightarrow R$  such that  $F$  is the base change of  $F_{BP}$  by  $f$ . If  $f(v_i) = 0$  for  $0 \leq i < n$ , then  $f$  induces a ring homomorphism  $\bar{f}: P(n)_* \rightarrow R$ . In this case we say that a  $p$ -typical formal group law  $F$  is of strict height at least  $n$ . Hence  $P(n)_*$  is the universal ring of  $p$ -typical formal group law of strict height at least  $n$ . We say that a ring homomorphism  $P(n)_* \rightarrow R$  is Landweber exact over  $P(n)$ , if the sequence  $v_n, v_{n+1}, \dots$  is regular in  $R$ . In this case, the functor  $R_* \otimes_{P(n)_*} P(n)_*(-)$  is a generalized homology theory by Landweber–Yagita exact functor theorem [12, 25], where  $R_*$  is the even-periodic commutative ring  $R[u^{\pm 1}]$  with  $|u| = -2$ . Furthermore, if  $R$  is a complete Noetherian local ring, then

$$R^*(X) = \varprojlim_{\Lambda(X)} (R_* \otimes_{P(n)_*} P(n)^*(Z))$$

is a generalized cohomology theory.

Let  $\mathbf{F}$  be a finite field which contains the finite fields  $\mathbf{F}_{p^n}$  and  $\mathbf{F}_{p^{n+1}}$ . Let  $E_n^*(-)$  be a variant of Morava  $E$ -theory whose coefficient ring is given by

$$E_{n*} = W(\mathbf{F}) \llbracket u_1, \dots, u_{n-1} \rrbracket [u^{\pm 1}],$$

where  $W(\mathbf{F})$  is the ring of Witt vectors with coefficients in  $\mathbf{F}$ . The grading is given by  $u_i = 0$  for  $1 \leq i < n$  and  $|u| = -2$ . Then the degree-0 formal group law  $F_n$  associated to  $E_n$  is a universal deformation of the Honda group law  $H_n$  of height  $n$  over  $\mathbf{F}$ . For  $0 \leq k \leq n$ , there is a commutative multiplicative cohomology theory  $(E_n/I_k)^*(-)$  whose coefficient ring is just  $E_{n*}/I_k$ , where  $I_k$  is the invariant prime ideal  $(p, u_1, \dots, u_{k-1})$ .

We define even-periodic graded commutative rings  $E_*$  and  $K_*$  as follows:

$$\begin{aligned} E_* &= \mathbf{F} \llbracket u_n \rrbracket [u^{\pm 1}], \\ K_* &= \mathbf{F} [w^{\pm 1}], \end{aligned}$$

where the gradings are given by  $|u_n| = 0, |w| = |u| = -2$ . The ring homomorphisms  $P(n)_* \rightarrow E_*$  given by  $v_n \mapsto u_n u^{-(p^n-1)}, v_{n+1} \mapsto u^{-(p^{n+1}-1)}, v_i \mapsto 0$  for  $i > n+1$ , and  $P(n)_* \rightarrow K_*$  given by  $v_n \mapsto w^{-(p^n-1)}, v_i \mapsto 0$  for  $i > n$ , make  $E_*$  and  $K_*$  Landweber exact  $P(n)_*$ -algebras, respectively. Hence

$$\begin{aligned} E^*(X) &= \varprojlim (E_* \otimes_{P(n)_*} P(n)^*(Z)), \\ K^*(X) &= \varprojlim (K_* \otimes_{P(n)_*} P(n)^*(Z)), \end{aligned}$$

define generalized cohomology theories. Note that there are no limit one problems since the degree-0 subrings are complete Noetherian local rings, respectively. The cohomology theory  $K^*(-)$  is a variant of Morava  $K$ -theory and the associated degree-0 formal group law is the Honda group law  $H_n$  of height  $n$  over  $\mathbf{F}$ . Since the cohomology theory  $E^*(-)$  is  $(E_{n+1}/I_n)^*(-)$ , the associated degree-0 formal group law is the base change of  $F_{n+1}$  to  $\mathbf{F}[[u_n]]$ .

We set

$$\begin{aligned} x_E &= 1 \otimes x \in E^0(\mathcal{X}), \\ y_E &= 1 \otimes y \in E^1(\mathcal{Y}), \\ x_K &= 1 \otimes x \in K^0(\mathcal{X}), \\ y_K &= 1 \otimes y \in K^1(\mathcal{Y}). \end{aligned}$$

Then we have

$$\begin{aligned} E^*(\mathcal{X}) &\cong E_*[[x_E]], \\ E^*(\mathcal{Y}) &\cong \Lambda(y_E) \otimes E_*[x_E]/(x_E^{p^n}), \\ K^*(\mathcal{X}) &\cong K_*[[x_K]], \\ K^*(\mathcal{Y}) &\cong \Lambda(y_K) \otimes K_*[x_K]/(x_K^{p^n}). \end{aligned}$$

## 2.2 Degeneration of formal groups

In this subsection we review some results in [23]. In this subsection  $p$  is any prime number. Let  $E_{n+1,0}$  be the degree-0 coefficient ring of the variant of Morava  $E$ -theory  $E_{n+1}$ :

$$E_{n+1,0} = W(\mathbf{F})[[u_1, \dots, u_n]].$$

The associated degree-0 formal group law  $F_{n+1}$  is a universal deformation of the Honda group laws  $H_{n+1}$  of height  $n+1$  over  $\mathbf{F}$ . The extended Morava stabilizer group  $G_{n+1} = \text{Gal} \ltimes S_{n+1}$  is the automorphism group of  $F_{n+1}$  in some generalized sense (cf Strickland [21] and Torii [23]), where  $\text{Gal}$  is the Galois group  $\text{Gal}(\mathbf{F}/\mathbf{F}_p)$  and  $S_{n+1}$  is the  $n$ -th Morava stabilizer group. Note that  $S_{n+1}$  is the automorphism group of  $H_{n+1}$



in the usual sense. The extended Morava stabilizer group  $G_{n+1}$  is a profinite group and acts on  $E_{n+1,0}$  continuously, where the topology of  $E_{n+1,0}$  is given by the adic-topology. Since the ideal  $I_n = (p, u_1, \dots, u_{n-1})$  of  $E_{n+1,0}$  is stable under the action,  $G_{n+1}$  also acts on the quotient ring  $E_{n+1,0}/I_n = \mathbf{F}[[u_n]]$  continuously.

We regard the formal group law  $F_{n+1}$  as being defined over  $\mathbf{F}[[u_n]]$  by the obvious base change. This situation is a kind of degeneration and a fundamental technique to study a degeneration is to investigate the monodromy representation. Let  $M = \mathbf{F}((u_n))$  be the quotient field of  $\mathbf{F}[[u_n]]$  and  $M^{\text{sep}}$  its separable closure. Then the height of  $F_{n+1}$  on  $M$  is  $n$ . Hence the fibre of  $F_{n+1}$  over  $M^{\text{sep}}$  is isomorphic to  $H_n$  since the isomorphism classes of formal group laws over a separably closed field are classified by their height. The monodromy representation of  $F_{n+1}$  around the closed point gives the following homomorphism:

$$\text{Gal}(M^{\text{sep}}/M) = \pi_1(M) \longrightarrow \text{Aut}(H_n) = S_n.$$

This homomorphism was studied by Gross in [6].

Let  $\Phi$  be an isomorphism over  $M^{\text{sep}}$  between  $F_{n+1}$  and  $H_n$ :

$$\Phi(F_{n+1}(X, Y)) = H_n(\Phi(X), \Phi(Y)).$$

Let  $L$  be a separable algebraic extension of  $M$  obtained by adjoining all the coefficients of  $\Phi(X)$ . Then the above homomorphism  $\text{Gal}(M^{\text{sep}}/M) \rightarrow S_n$  induces an isomorphism  $\text{Gal}(L/M) \xrightarrow{\cong} S_n$ , and this extends to an isomorphism  $\text{Gal}(L/\mathbf{F}_p((u_n))) \cong G_n$ . Let  $\mathcal{G}$  be the semidirect product  $\text{Gal} \ltimes (S_n \times S_{n+1})$ . Then  $\mathcal{G}$  is a profinite group, and contains  $G_n$  and  $G_{n+1}$  as closed subgroups.

The following theorem is a main point of [23].

**Theorem 2.4** [23, Section 2.4] *The profinite group  $\mathcal{G}$  acts on the formal group law  $(F_{n+1}, L)$  in generalized sense. The action of the subgroup  $G_{n+1}$  is an extension of the action on  $(F_{n+1}, \mathbf{F}[[u_n]])$ . The action of the subgroup  $G_n$  on  $(F_{n+1}, L)$  is the action of Galois group on  $L$  and the trivial action on  $F_{n+1}$ . Under the isomorphism  $\Phi: F_{n+1} \xrightarrow{\cong} H_n$ , the induced action of  $\mathcal{G}$  on  $(H_n, L)$  is encoded as the following two commutative diagrams. For  $g \in G_{n+1}$ , there is a commutative diagram*

$$(3) \quad \begin{array}{ccc} F_{n+1} & \xrightarrow{t_E(g)} & F_{n+1}^g \\ \Phi \downarrow & & \downarrow \Phi^g \\ H_n & \xrightarrow{=} & H_n^g \end{array}$$

where  $t_E(g)(X)$  is an isomorphism from  $F_{n+1}$  to  $F_{n+1}^g$  corresponding to  $g$ . For  $h \in G_n$ , there is a commutative diagram

$$(4) \quad \begin{array}{ccc} F_{n+1} & \xrightarrow{=} & F_{n+1}^h \\ \Phi \downarrow & & \downarrow \Phi^h \\ H_n & \xrightarrow{t_K(h)} & H_n^h, \end{array}$$

where  $t_K(h)(X)$  is the automorphism of  $H_n$  corresponding to  $h$ .

### 2.3 The generalized Chern character

In this subsection we review the generalized Chern character  $\Theta$  constructed in [22].

The co-operation ring  $P(n)_*(P(n))$  is isomorphic to

$$P(n)_*[t_1, t_2, \dots] \otimes \Lambda(a_{(0)}, \dots, a_{(n-1)})$$

as a left  $P(n)_*$ -algebra, where  $|t_i| = 2(p^i - 1)$  and  $|a_{(i)}| = 2p^i - 1$ . In particular,  $P(n)_*(P(n))$  is a free  $P(n)_*$ -module. Hence  $(P(n)_*, P(n)_*(P(n)))$  is a Hopf algebroid over  $\mathbf{F}_p$ . By formalism of Boardman [2], there is a natural  $P(n)_*(P(n))$ -comodule structure on the completion  $P(n)^*(X)^\wedge$  with respect to the profinite topology:

$$\rho: P(n)^*(X)^\wedge \longrightarrow P(n)_*(P(n)) \hat{\otimes}_{P(n)_*} P(n)^*(X)^\wedge.$$

The set of  $\mathbf{F}_p$ -algebra homomorphisms from  $P(n)_*(P(n))/(a_{(0)}, \dots, a_{(n-1)})$  to an even-periodic  $\mathbf{F}_p$ -algebra  $R_*$  is naturally identified with the set of triples  $(F, f, G)$ , where  $F$  and  $G$  are  $p$ -typical formal group laws over  $R_0$  with strict height at least  $n$ , and  $f$  is an isomorphism between them. Let  $L_*$  be an even-periodic  $E_*$ -algebra  $L[u^{\pm 1}]$ . By [Theorem 2.4](#) and the above moduli interpretation of the ring  $P(n)_*(P(n))/(a_{(0)}, \dots, a_{(n-1)})$ , there is a ring homomorphism  $\theta: P(n)_*(P(n)) \rightarrow P(n)_*(P(n))/(a_{(0)}, \dots, a_{(n-1)}) \rightarrow L_*$  such that the following diagram commutes:

$$\begin{array}{ccc} P(n)_* & \xrightarrow{H_n} & K_* \\ \eta_R \downarrow & & \downarrow \\ P(n)_*(P(n)) & \xrightarrow{\theta} & L_* \\ \eta_L \uparrow & & \uparrow \\ P(n)_* & \xrightarrow{F_{n+1}} & E_* \end{array}$$

That is,  $\theta$  corresponds to the triple  $(F_{n+1}, \Phi, H_n)$  over  $L$ . For  $Z \in \Lambda(X)$ , by extending the natural ring homomorphism

$$\begin{aligned} P(n)^*(Z) &\xrightarrow{\rho} P(n)_*(P(n)) \otimes_{P(n)_*} P(n)^*(Z) \\ &\xrightarrow{\theta \otimes 1} L_* \otimes_{P(n)_*} P(n)^*(Z) \\ &\cong L^*(Z) \end{aligned}$$

to  $E^*(Z) = E_* \otimes_{P(n)_*} P(n)^*(Z) \rightarrow L^*(Z)$ , we obtain a multiplicative natural transformation

$$(5) \quad \Theta : E^*(X) \longrightarrow L^*(X),$$

which we call the generalized Chern character.

The following lemma is easily checked.

**Lemma 2.5**  $\Theta(x_E) = \Phi(x_K)$  and  $\Theta(y_E) = 1 \otimes y_K$ .

### 3 Complete Hopf algebroids

#### 3.1 Complete modules

Let  $k$  be a commutative ring. We say that  $(M, \{F^\lambda M\}_{\lambda \in \Lambda})$  is a filtered  $k$ -module if  $M$  is a  $k$ -module and  $\{F^\lambda M\}_{\lambda \in \Lambda}$  is a family of  $k$ -submodules indexed by a (small) filtered category  $\Lambda$ . Then  $M$  can be given a linear topology. We denote by  $\mathbf{FMod}_k$  the category of filtered  $k$ -modules and continuous homomorphisms. A filtered  $k$ -module  $(M, \{F^\lambda M\}_{\lambda \in \Lambda})$  is said to be complete Hausdorff if the canonical homomorphism

$$M \rightarrow \varprojlim_{\Lambda} M/F^\lambda M$$

is an isomorphism. We denote by  $\mathbf{FMod}_k^c$  the full subcategory of  $\mathbf{FMod}_k$  whose objects are complete Hausdorff. We say that  $(M, \{F^\lambda M\}_{\lambda \in \Lambda}) \in \mathbf{FMod}_k^c$  is a profinite  $k$ -module if  $M/F^\lambda M$  is a finitely generated  $k$ -module for all  $\lambda \in \Lambda$ . We denote by  $\mathbf{ProFG}_k$  the full subcategory of  $\mathbf{FMod}_k^c$  whose objects are profinite.

Since  $\mathbf{FMod}_k^c$  is a symmetric monoidal category with tensor product  $\widehat{\otimes}_k$  and unit object  $k$ , we can define commutative monoid objects in  $\mathbf{FMod}_k^c$ , that is, complete commutative  $k$ -algebras. We denote by  $\mathbf{FAlg}_k^c$  the category of complete commutative  $k$ -algebras. For  $R \in \mathbf{FAlg}_k^c$ , we can define an  $R$ -module in  $\mathbf{FMod}_k^c$ , and we denote by  $\mathbf{FMod}_R^c$  the category of  $R$ -modules. For  $R_1 \rightarrow R_2 \in \mathbf{FAlg}_k^c$ , there is a base change functor

$$(-) \widehat{\otimes}_{R_1} R_2 : \mathbf{FMod}_{R_1}^c \longrightarrow \mathbf{FMod}_{R_2}^c.$$

If  $R$  is a complete Noetherian local  $k$ -algebra with maximal ideal  $\mathfrak{m}$ , then  $R$  with  $\mathfrak{m}$ -adic filtration can be regarded as an object in  $\mathbf{FAlg}_k^c$ . We denote by  $R^c$  the  $k$ -module  $R$  with  $\mathfrak{m}$ -adic filtration, and simply by  $R$  the  $k$ -module  $R$  with trivial filtration  $\{0\}$ . Note that the base change  $M \widehat{\otimes}_R R^c$  for  $M \in \mathbf{FMod}_R^c$  is given by

$$\varprojlim (F^\lambda M + \mathfrak{m}^i M)$$

with inverse limit topology. Since  $M$  is isomorphic to the inverse limit of  $M/\mathfrak{m}^i M$  for a finitely generated  $R$ -module  $M$ , we see that  $M \widehat{\otimes}_R R^c \cong M$  as (abstract)  $R$ -modules for  $M \in \mathbf{ProFG}_R$ .

**Example 3.1** Let  $h^*(-)$  be a generalized cohomology theory and  $X$  a spectrum. We defined the profinite filtration on  $h^*(X)$  in Remark 2.3. If  $h^*(-)$  is even-periodic and the degree-0 coefficient ring  $h_0$  is a complete Noetherian local ring, then  $h^*(X)$  is a complete Hausdorff profinite  $h_*$ -module. Hence the cohomology theory  $h^*(-)$  gives a functor from the stable homotopy category to  $\mathbf{ProFG}_{h_*}$ .

**Lemma 3.2** Let  $M \in \mathbf{FMod}_k^c$  and  $\underline{M}$  the underlying  $k$ -module. If  $M \in \mathbf{FMod}_R^c$ , then  $\underline{M}$  is an  $R$ -module in the usual sense.

**Proof** The map  $R \otimes M \rightarrow R \widehat{\otimes} M \rightarrow M$  gives an  $R$ -module structure on  $\underline{M}$ .  $\square$

**Lemma 3.3** If  $M \in \mathbf{FMod}_R^c$ , then for any open  $k$ -submodule  $M_\lambda$  there is an open  $R$ -submodule  $N$  such that  $N \subset M_\lambda$ .

**Proof** The fact that  $M \in \mathbf{FMod}_R^c$  implies that the map  $R \widehat{\otimes} M \rightarrow M \rightarrow M/M_\lambda$  factors through  $R \otimes M/M_\mu$  for some open  $k$ -submodule  $M_\mu$ . Hence  $R \cdot M_\mu \subset M_\lambda$ . We take  $N$  as  $R \cdot M_\mu$ . Since  $M_\mu \subset N$ ,  $N$  is an open  $R$ -submodule. This completes the proof.  $\square$

**Corollary 3.4** Let  $M \in \mathbf{FMod}_R^c$ . There is a fundamental (open) neighborhood system at 0 consisting of  $R$ -submodules.

**Corollary 3.5** Let  $M$  and  $N$  be objects in  $\mathbf{FMod}_R^c$ . Then

$$M \widehat{\otimes}_R N \cong \varprojlim_{\lambda, \mu} (M/F^\lambda M) \otimes_R (N/F^\mu N),$$

where  $\{F^\lambda M\}_\lambda$  and  $\{F^\mu N\}_\mu$  are families of all open  $R$ -submodules of  $M$  and  $N$ , respectively.

**Lemma 3.6** *Let  $M \in \text{FMod}_R^c$ . Then  $M$  is an  $R^c$ -module compatible with given  $R$ -module structure if and only if for any open  $R$ -submodule  $N$  there is a nonnegative integer  $i$  such that  $\mathfrak{m}^i M \subset N$ .*

**Proof** If  $M$  is an  $R^c$ -module compatible with given  $R$ -module structure, then there is a continuous map  $R^c \widehat{\otimes}_R M \rightarrow M$ , which makes  $M$  an  $R^c$ -module. Then the map  $R^c \widehat{\otimes}_R M \rightarrow M \rightarrow M/N$  factors through  $R/\mathfrak{m}^i \otimes_R M/N'$  for some  $i$  and some open  $R$ -submodule  $N'$ . This implies that  $\mathfrak{m}^i M \subset \mathfrak{m}^i M + N' \subset N$ .

If for any open  $R$ -submodule  $N$  there is  $i$  such that  $\mathfrak{m}^i M \subset N$ , then there are compatible maps  $R^c \widehat{\otimes}_R M \rightarrow R/\mathfrak{m}^i \otimes_R M/N \rightarrow M/N$ , which induce a continuous map  $R^c \widehat{\otimes}_R M \rightarrow M$ . This map defines an  $R^c$ -module structure on  $M$  compatible with given  $R$ -module structure.  $\square$

**Lemma 3.7** *Let  $M \in \text{FMod}_k^c$ . If a profinite group  $G$  acts on  $M$  continuously as  $k$ -module homomorphisms, then for any open submodule  $M_\lambda$  there is an open submodule  $N$  such that  $G \cdot N \subset M_\lambda$ .*

**Proof** For any  $g \in G$ , there are an open submodule  $N_g$  of  $M$  and an open neighborhood  $U_g$  of  $g$  such that  $U_g \cdot N_g \subset M_\lambda$ . Since  $G$  is compact,  $G = U_{g_1} \cup \cdots \cup U_{g_n}$ . Take an open submodule  $N$  such that  $N \subset N_{g_1} \cap \cdots \cap N_{g_n}$ . Then for any  $g \in G$ ,  $g \in U_{g_i}$  for some  $i$ , and for any  $x \in N \subset N_{g_i}$ ,  $g \cdot x \in U_{g_i} \cdot N_{g_i} \subset M_\lambda$ . Hence we obtain that  $G \cdot N \subset M_\lambda$ .  $\square$

**Corollary 3.8** *Let  $M \in \text{FMod}_k^c$ . If a profinite group  $G$  acts on  $M$  continuously as  $k$ -modules homomorphisms, then for any open submodule  $M_\lambda$ , there is an open  $G$ -submodule  $N$  such that  $N \subset M_\lambda$ .*

**Proof** By Lemma 3.7, there is an open submodule  $N'$  such that  $G \cdot N' \subset M_\lambda$ . Let  $N$  be the submodule generated by  $G \cdot N'$ . Then  $N$  is a  $G$ -submodule and  $N \subset M_\lambda$ . Since  $N' \subset N$ ,  $N$  is an open submodule. This completes the proof.  $\square$

**Corollary 3.9** *Let  $M \in \text{FMod}_k^c$  and  $G$  a profinite group. Suppose that  $G$  acts on  $M$  continuously as  $k$ -module homomorphisms. There is a fundamental (open) neighborhood system at 0 consisting of  $G$ -submodules.*

**Theorem 3.10** *Let  $G$  be a profinite group acting on  $R^c$  continuously as  $k$ -algebra homomorphisms, and  $M$  a complete twisted  $R^c$ - $G$ -module. For any open  $R$ -submodule  $M_\lambda$ , there is an open  $R$ - $G$ -submodule  $N$  such that  $N \subset M_\lambda$ .*

**Proof** By Corollary 3.8, there is an open  $k$ - $G$ -submodule  $N'$  such that  $N' \subset M_\lambda$ . Let  $N$  be the  $R$ -submodule generated by  $N'$ . Then  $N$  is an open  $G$ -submodule such that  $N \subset M_\lambda$ .  $\square$

**Corollary 3.11** *Let  $M$  be a complete twisted  $R^c$ - $G$ -module. Then there is a fundamental (open) neighborhood system at 0 consisting of  $R$ - $G$ -submodules.*

### 3.2 Complete Hopf algebroids and complete precomodules

Let  $A$  and  $\Gamma$  be objects in  $\mathbf{FAlg}_k^c$ . We suppose that there are maps in  $\mathbf{FAlg}_k^c$ :

$$\begin{aligned}\eta_R: A &\longrightarrow \Gamma, \\ \eta_L: A &\longrightarrow \Gamma, \\ \chi: \Gamma &\longrightarrow \Gamma, \\ \varepsilon: \Gamma &\longrightarrow A.\end{aligned}$$

If the maps  $(\eta_R, \eta_L, \chi, \varepsilon)$  satisfy the usual Hopf algebroid relations [18, Appendix 1], then we say that the pair  $(A, \Gamma)$  is a complete Hopf algebroid over  $k$ . A Hopf algebroid is a complete Hopf algebroid with discrete topology. Since  $P(n)_*(P(n))$  is free over  $P(n)_*$ ,  $P(n)_*(P(n))$  is a Hopf algebroid, hence, a complete Hopf algebroid over  $\mathbf{F}_p$ .

Let  $A \rightarrow B$  be a map in  $\mathbf{FAlg}_k^c$ . We set

$$\Gamma_B := B \hat{\otimes}_A \Gamma \hat{\otimes}_A B.$$

Then  $(B, \Gamma_B)$  is a complete Hopf algebroid over  $k$  as usual.

**Example 3.12** Let  $h^*(-)$  be an even-periodic Landweber exact theory over  $P(n)$  such that  $h_0$  is a complete Noetherian local ring. Put

$$\Gamma(h) = h_*^c \hat{\otimes}_{P(n)_*} P(n)_*(P(n)) \hat{\otimes}_{P(n)_*} h_*^c.$$

Then  $(h_*^c, \Gamma(h))$  is a complete Hopf algebroid over  $\mathbf{F}_p$ .

An object  $M \in \mathbf{FMod}_A^c$  is said to be a complete  $\Gamma$ -comodule if there is a continuous map  $\rho: M \rightarrow \Gamma \hat{\otimes}_A M$  such that obvious co-associativity and co-unity diagrams commute.

**Definition 3.13** Let  $R$  be a complete Noetherian local ring, and  $(R^c, \Gamma)$  a complete Hopf algebroid over  $k$ . An object  $M \in \mathbf{FMod}_R^c$  is said to be a complete  $\Gamma$ -precomodule if there is a continuous map

$$\rho: M \longrightarrow \Gamma \hat{\otimes}_R M$$

such that the following two conditions are satisfied:

- (1) For any open  $R$ -submodule  $M_\lambda$  of  $M$ , there is an open  $R$ -submodule  $M_\mu$  such that the map  $M \xrightarrow{\rho} \Gamma \widehat{\otimes}_R M \rightarrow \Gamma \widehat{\otimes}_R M / M_\lambda$  factors through  $M / M_\mu$ . When the above condition is satisfied,  $\rho$  induces a continuous map  $\rho^c : M^c \rightarrow \Gamma \widehat{\otimes}_{R^c} M^c$ .
- (2) The continuous map  $\rho^c$  makes  $M^c$  a complete  $\Gamma$ -comodule.

Furthermore, if  $M$  is a complete Hausdorff commutative  $R$ -algebra and  $\rho^c$  is a map of complete  $R^c$ -algebras, then  $M$  is said to be a complete  $\Gamma$ -precomodule algebra.

For  $Z$  finite, the coaction map  $P(n)^*(Z) \rightarrow P(n)_*(P(n)) \otimes_{P(n)_*} P(n)^*(Z)$  induces a natural continuous map

$$h^*(Z) \longrightarrow \Gamma(h) \widehat{\otimes}_{h_*} h^*(Z).$$

**Proposition 3.14** Let  $h^*(-)$  be an even-periodic Landweber exact theory over  $P(n)$  such that  $h_0$  is a complete Noetherian local ring. Then  $h^*(Z)$  has a natural  $\Gamma(h)$ -precomodule structure for finite  $Z$ . Furthermore, if  $Z$  is a finite CW-complex, then  $h^*(Z)$  is a  $\Gamma(h)$ -precomodule algebra.

**Proof** Since  $h^*(Z)$  is discrete if  $Z$  is finite, the condition (1) is trivial. Actually,  $h^*(Z)$  is a  $h_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} h_*$ -comodule. Hence  $h^*(Z)^c = h_*^c \widehat{\otimes}_{h_*} h^*(Z)$  is  $\Gamma(h)$ -comodule. If  $Z$  is a finite CW-complex, it is easy to see that  $h^*(Z)$  is a  $\Gamma(h)$ -precomodule algebra.  $\square$

## 4 Complete Hopf algebroid $C(G, R_*^c)$

### 4.1 The Hopf algebroid structure of $C(G, R_*^c)$

Let  $k$  be a commutative ring and  $R_*$  an even-periodic graded commutative  $k$ -algebra such that the degree-0 subring  $R_0$  is a complete local ring with maximal ideal  $\mathfrak{m}_0$ . We denote by  $R_*^c$  a graded topological ring  $R_*$  with  $\mathfrak{m}$ -adic topology, where  $\mathfrak{m} = \mathfrak{m}_0 R$ . Let  $G$  be a profinite group, which continuously acts on  $R_*^c$  as  $k$ -algebra automorphisms from the right. Let  $C = C(G, R_*^c)$  be the set of all continuous maps from  $G$  to  $R_*^c$ . Then  $C$  is an even-periodic commutative ring from the ring structure on  $R_*^c$ . It is known that the pair  $(R_*^c, C)$  is a graded complete Hopf algebroid over  $k$ . In this section we describe the structure of  $(R_*^c, C)$  (cf [9, Section 6.3]).

First, note that there is an isomorphism of commutative rings

$$C = C(G, R_*^c) \cong \varprojlim_i C(G, R_*/\mathfrak{m}^i),$$

where  $C(G, R_*/\mathfrak{m}^i)$  is the ring of all continuous map from  $G$  to the discrete ring  $R_*/\mathfrak{m}^i$ . We give the inverse limit topology to  $C$ , where  $C(G, R_*/\mathfrak{m}^i)$  is discrete. The projection  $R_*^c \times G \rightarrow R_*^c$  gives a continuous ring homomorphism  $\eta_R: R_*^c \rightarrow C$ . By the ring homomorphism  $k \rightarrow R_*^c \xrightarrow{\eta_R} C$ , we regard  $C$  as a commutative  $k$ -algebra. The action  $R_*^c \times G \rightarrow R_*^c$  gives a continuous ring homomorphism  $\eta_R: R_*^c \rightarrow C$ , which is a  $k$ -algebra homomorphism.

Let  $C(G \times G, R_*^c)$  be the ring of all continuous maps from  $G \times G$  to  $R_*^c$ . Then  $C(G \times G, R_*^c)$  is a complete commutative  $k$ -algebra as in  $C$ .

Let  $G$  be a profinite group. We denote by  $C(G, M)$  the set of all continuous maps from  $G$  to  $M \in \text{FMod}_k^c$ . Then it can be given a  $k$ -module structure on  $C(G, M)$  from the  $k$ -module structure on  $M$ . There is an isomorphism of  $k$ -modules

$$C(G, M) \cong \varprojlim_N \varinjlim_U F(G/U, M/N),$$

where  $F(G/U, M/N)$  is the set of all maps from  $G/U$  to  $M/N$ ,  $N$  ranges over all open submodules of  $M$ , and  $U$  ranges over all open normal subgroup of  $G$ . We regard  $C(G, M)$  as an object in  $\text{FMod}_k^c$  by inverse limit topology.

**Lemma 4.1** *For a profinite group  $G$  and  $M \in \text{FMod}_k^c$ , there is a natural isomorphism in  $\text{FMod}_k^c$ :*

$$C(G, k) \widehat{\otimes}_k M \cong C(G, M).$$

**Proof** We have an isomorphism

$$C(G, k) \widehat{\otimes} M \cong \varprojlim_N \varinjlim_U F(G/U, k) \otimes M/N.$$

Since  $G/U$  is a finite set,  $F(G/U, k) \otimes M/N \cong F(G/U, M/N)$ . Hence we see that  $C(G, k) \widehat{\otimes} M \cong \varprojlim_N \varinjlim_U F(G/U, M/N) \cong C(G, M)$ .  $\square$

Let  $m: C \times C \rightarrow C(G \times G, R_*^c)$  be a map given by  $m(\alpha, \beta)(g_1, g_2) = \alpha(g_1)^{g_2} \beta(g_2)$  for  $\alpha, \beta \in C, g_1, g_2 \in G$ . The map  $m$  induces an isomorphism of complete commutative  $k$ -algebras:

$$C \widehat{\otimes}_{R_*^c} C \xrightarrow{\cong} C(G \times G, R_*^c).$$

We define a map  $\psi$  by

$$\psi: C \xrightarrow{\tilde{\psi}} C(G \times G, R_*^c) \cong C \widehat{\otimes}_{R_*^c} C,$$

where  $\tilde{\psi}$  is the map induced by the multiplication  $G \times G \rightarrow G$ . Then we can check that  $\psi$  is a continuous  $k$ -algebra homomorphism.



Let  $\chi: C \rightarrow C$  be the map given by  $\chi(\alpha)(g) = \alpha(g^{-1})^g$  for  $\alpha \in C, g \in G$ . Then it is easy to see that  $\chi$  is a continuous  $k$ -algebra automorphism. Let  $\varepsilon: C \rightarrow R_*^c$  be the map given by  $\varepsilon(\alpha) = \alpha(e)$  for  $\alpha \in C$ , where  $e$  is the identity element of  $G$ . Then it is also easy to see that  $\varepsilon$  is a continuous  $k$ -algebra homomorphism.

**Theorem 4.2** (cf Hovey [9, Section 6.3]) *The pair  $(R_*^c, C)$  with  $(\eta_R, \eta_L, \psi, \chi, \varepsilon)$  is a graded complete Hopf algebroid over  $k$ .*

**Remark 4.3** Let  $C(G, k)$  be the ring of all continuous maps from  $G$  to  $k$ . There is an isomorphism  $C \cong C(G, k) \hat{\otimes}_k R_*^c$  of complete  $k$ -algebras by Lemma 4.1, and  $C(G, k)$  is a Hopf algebra over  $k$  by Theorem 4.2. The right action of  $G$  on  $R_*^c$  gives  $R_*^c$  a graded (left)  $C(G, k)$ -comodule algebra structure. Let  $\rho: R_*^c \rightarrow C(G, k) \hat{\otimes}_k R_*^c$  be the comodule algebra structure map. In this situation we can construct a split Hopf algebroid  $(R_*^c, C(G, k) \hat{\otimes}_k R_*^c)$ . In fact,  $\rho = \eta_L$  under the above isomorphism, and the graded complete Hopf algebroid  $(R_*^c, C)$  is isomorphic to  $(R_*^c, C(G, k) \hat{\otimes}_k R_*^c)$ .

## 4.2 Twisted modules

In this subsection we show that there is an equivalence of symmetric monoidal categories between the category of complete  $C$ -comodules and the category of complete twisted  $R_*^c$ - $G$ -modules.

**Definition 4.4** A complete Hausdorff filtered  $R_*^c$ -module  $M$  is said to be a complete twisted (right)  $R_*^c$ - $G$ -module if  $G$  acts on  $M$  continuously (from the right) such that  $(am)g = a^g \cdot (m)g$  for all  $m \in M, a \in R_*, g \in G$ .

**Remark 4.5** The category of complete twisted  $R_*^c$ - $G$ -modules is a symmetric monoidal category under complete tensor product  $\hat{\otimes}_{R_*^c}$  and unit object  $R_*^c$ .

**Definition 4.6** A complete Hausdorff filtered  $R_*^c$ -module  $M$  is said to be a complete (left)  $C$ -comodule if there is a continuous left  $R_*^c$ -module homomorphism  $\rho_M: M \rightarrow C \hat{\otimes}_{R_*^c} M$ , which makes co-associativity and co-unity diagrams commute.

**Remark 4.7** The category of complete  $C$ -comodules is a symmetric monoidal category under complete tensor product  $\hat{\otimes}_{R_*^c}$  and unit object  $R_*^c$ .

**Lemma 4.8** For a complete (left)  $C$ -comodule  $M$ , there is a natural complete twisted (right)  $R_*^c$ - $G$ -module structure on  $M$ .

**Proof** We denote by  $\text{ev}(g): C \rightarrow R_*^c$  the evaluation map at  $g \in G$ . Then the map

$$M \longrightarrow C \hat{\otimes}_{R_*^c} M \xrightarrow{\text{ev}(g) \otimes 1} R_*^c \hat{\otimes}_{R_*^c} M \cong M$$

defines a twisted  $R_*$ - $G$ -module structure on  $M$ . Hence it is sufficient to show that the action map  $M \times G \rightarrow M$  is continuous.

Let  $N$  be an open  $R$ -submodule of  $M$ . Then  $\mathfrak{m}^i M \subset N$  for some  $i$ , since  $M$  is an  $R_*^c$ -module. In this case,  $C(G, R/\mathfrak{m}^i) \otimes_R M/N \cong C(G, M/N)$ . Then there is an open  $R$ -submodule  $N'$ , which makes the following diagram commute:

$$\begin{array}{ccc} M & \longrightarrow & C \hat{\otimes} M \\ \downarrow & & \downarrow \\ M/N' & \longrightarrow & C(G, M/N). \end{array}$$

We note that for any element of  $M/N'$  the image under the bottom arrow factors through  $F(G/U, M/N)$  for some open normal subgroup  $U$  of  $G$ . The above commutative diagram gives us the following commutative diagram:

$$\begin{array}{ccc} M \times G & \longrightarrow & M \\ \downarrow & & \downarrow \\ M/N' \times G & \longrightarrow & M/N. \end{array}$$

By the above remark, the bottom arrow is continuous. Hence top arrow is also continuous. This completes the proof.  $\square$

**Lemma 4.9** *Let  $M$  be an  $R_*^c$ -module and  $G$  a profinite group. Then there is an isomorphism of left  $R_*^c$ -modules*

$$C \hat{\otimes}_{R_*^c} M \cong C(G, M),$$

where the left  $R_*^c$ -module structure on  $C \hat{\otimes} M$  comes from  $\eta_L$  of  $C$ , and the left  $R_*^c$ -module structure on  $C(G, M)$  is given by  $(r \cdot f)(g) = r^g \cdot f(g)$  for  $r \in R$ ,  $f \in C(G, M)$ ,  $g \in G$ .

**Proof** There is an isomorphism  $C \cong C(G, k) \hat{\otimes}_k R_*^c$  by Lemma 4.1. Hence  $C \hat{\otimes}_{R_*^c} M$  is isomorphic to  $C(G, k) \hat{\otimes}_k M$ . Then we have

$$\begin{aligned} C(G, k) \hat{\otimes}_k M &\cong \varprojlim_N \varinjlim_U F(G/U, k) \otimes_k M/N \\ &\cong \varprojlim_N \varinjlim_U F(G/U, M/N) \\ &\cong \varprojlim_N C(G, M/N) \\ &\cong C(G, M). \end{aligned}$$

It is easy to check that this isomorphism respects the left  $R_*^c$ -module structures.  $\square$

**Lemma 4.10** *For a complete twisted (right)  $R_*^c$ - $G$ -module  $M$ , there is a natural complete (left)  $C$ -comodule structure on  $M$ .*

**Proof** The  $G$ -module structure map  $M \times G \rightarrow M$  gives a map  $M \rightarrow C(G, M)$ . By Lemma 4.9, we obtain a map  $M \rightarrow C \hat{\otimes} M$ . If this map is continuous, it is easy to check that it defines a complete  $C$ -comodule structure on  $M$ . Hence it is sufficient to show that  $M \rightarrow C(G, M)$  is continuous. For any open submodule  $N$  of  $M$  and  $g \in G$ , there are open submodule  $N_g$  of  $M$  and an open neighborhood  $U_g$  of  $g$  such that  $N_g \cdot U_g \subset N$ . Since  $G$  is compact,  $G = U_{g_1} \cup \cdots \cup U_{g_n}$ . Let  $N'$  be an open submodule such that  $N' \subset N_{g_1} \cap \cdots \cap N_{g_n}$ . Then  $N' \cdot G \subset N$ . Hence the map  $M \times G \rightarrow M \rightarrow M/N$  factors through  $M/N' \times G$ . This implies that the map  $M \rightarrow C(G, M) \rightarrow C(G, M/N)$  factors through  $M/N'$ . Hence  $M \rightarrow C(G, M)$  is continuous. This completes the proof.  $\square$

**Theorem 4.11** *There is an equivalence of symmetric monoidal categories between the category of complete twisted (right)  $R_*^c$ - $G$ -modules and the category of complete (left)  $C$ -comodules.*

**Proof** By Lemma 4.8 and Lemma 4.10, there is an equivalence of categories between the category of complete  $C$ -comodules and the category of complete twisted  $R_*^c$ - $G$ -modules. It is easy to check that this equivalence respects the symmetric monoidal structures.  $\square$

### 4.3 Remark on twisted modules

In this subsection we let  $G = \text{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \rtimes S_n$ . Usually,  $G$  is called the  $n$ -th extended Morava stabilizer group, and it is important to study the category of complete twisted  $\mathbf{F}_{p^n}$ - $G$ -modules. In this subsection we compare the category of complete twisted  $\mathbf{F}_{p^n}$ - $G$ -modules and the category of complete twisted  $\mathbf{F}$ - $G_n$ -modules.

There is an exact sequence:

$$1 \rightarrow \text{Gal}(\mathbf{F}/\mathbf{F}_{p^n}) \longrightarrow G_n \longrightarrow G \rightarrow 1.$$

Hence, for a twisted  $\mathbf{F}$ - $G_n$ -module  $M$ , the submodule  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  invariant over  $\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})$  is a twisted  $\mathbf{F}_{p^n}$ - $G$ -module. Conversely, for a twisted  $\mathbf{F}_{p^n}$ - $G$ -module  $N$ , we can give  $\mathbf{F} \otimes_{\mathbf{F}_{p^n}} N$  an obvious twisted  $\mathbf{F}$ - $G_n$ -module structure.

**Lemma 4.12** For a finite dimensional twisted  $\mathbf{F}$ – $G_n$ –module  $M$ ,  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  is a finite dimensional twisted  $\mathbf{F}_{p^n}$ – $G$ –module, and  $\mathbf{F} \otimes_{\mathbf{F}_{p^n}} M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  is isomorphic to  $M$  as a twisted  $\mathbf{F}$ – $G_n$ –module.

**Proof** Let  $m = \dim_{\mathbf{F}} M$ . We obtain an isomorphism  $M \cong \mathbf{F} \otimes_{\mathbf{F}_{p^n}} M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  as twisted  $\mathbf{F}$ – $\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})$ –modules, since we have  $H^1(\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n}); \text{GL}_m(\mathbf{F})) = \{1\}$  (cf Serre [20, Proposition X.1.3]). By the above exact sequence,  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  is a twisted  $\mathbf{F}_{p^n}$ – $G$ –module, and we see that this is an isomorphism of twisted  $\mathbf{F}$ – $G_n$ –modules.  $\square$

**Remark 4.13** Since the cardinality of  $M$  is finite, the action of  $G_n$  on  $M$  and the action of  $G$  on  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  are continuous by Lemma 4.20.

Let  $M$  be a profinite twisted  $\mathbf{F}$ – $G_n$ –module. By Corollary 3.11, we can take a fundamental neighborhood system  $\{F^\lambda M\}$  at 0 consisting of open  $\mathbf{F}$ – $G_n$ –submodules. Then

$$M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})} \cong \varprojlim (M/F^\lambda M)^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})},$$

and  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  is a profinite twisted  $\mathbf{F}_{p^n}$ – $G$ –module with filtration  $F^\lambda(M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})})$ , where  $F^\lambda(M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})})$  is the kernel of the map  $M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})} \rightarrow (M/F^\lambda M)^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ . Conversely, for a profinite twisted  $\mathbf{F}_{p^n}$ – $G$ –module  $N$ , we can give  $\widehat{\mathbf{F} \otimes_{\mathbf{F}_{p^n}} N}$  an obvious profinite twisted  $\mathbf{F}$ – $G_n$ –module structure. By Lemma 4.12 and Remark 4.13, we obtain the following proposition.

**Proposition 4.14** The functor  $M \mapsto M^{\text{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$  gives an equivalence of symmetric monoidal categories between the category of profinite twisted  $\mathbf{F}$ – $G$ –modules and the category of profinite twisted  $\mathbf{F}_{p^n}$ – $G$ –modules. The quasi-inverse of this functor is given by  $N \mapsto \widehat{\mathbf{F} \otimes_{\mathbf{F}_{p^n}} N}$ .

By Proposition 4.14, there is no essential difference between profinite twisted  $\mathbf{F}$ – $G_n$ –modules and profinite twisted  $\mathbf{F}_{p^n}$ – $G$ –modules.

#### 4.4 Reformulation

In this section we reformulate the results in [22]. Set

$$\begin{aligned} C_{E_*} &= C(G_{n+1}, E_*^c), \\ C_{K_*} &= C(G_n, K_*). \end{aligned}$$

Then  $(E_*^c, C_{E_*})$  and  $(K_*, C_{K_*})$  are graded complete Hopf algebroids over  $\mathbf{F}_p$  by Theorem 4.2.

Let  $M$  be a profinite  $C_{E_*}$ -precomodule. Then  $M^c$  is a complete twisted  $E_*^c$ - $G_{n+1}$ -module by [Lemma 4.8](#). Note that  $M^c = M$  as an abstract  $E_*$ -module. Since the  $G_{n+1}$  action on  $L$  is compatible with the  $G_{n+1}$ -action on  $M$ ,  $G_{n+1}$  acts on  $M \widehat{\otimes}_{E_*} L_*$ , where  $L_* = L[u^{\pm 1}]$  is regarded as a discrete module. We define  $\mathcal{F}(M)$  to be the  $S_{n+1}$ -invariant submodule of  $L_* \widehat{\otimes}_{E_*} M$ :

$$\mathcal{F}(M) = H^0(S_{n+1}; L_* \widehat{\otimes}_{E_*} M).$$

We regard  $K_* = \mathbf{F}[w^{\pm 1}]$  as a subring of  $L_* = L[u^{\pm 1}]$  by  $w = \Phi_0^{-1}u$ . The following lemma was proved in [\[22, Lemma 4.2\]](#).

**Lemma 4.15**  $H^0(S_{n+1}; L_*) = K_*$ .

Note that  $E_*$  with discrete topology is a  $C_{E_*}$ -precomodule.

**Corollary 4.16**  $\mathcal{F}(E_*) = K_*$ .

**Lemma 4.17** Let  $M_L$  be a finite dimensional twisted  $L_*$ - $G_{n+1}$ -module. Then the dimension of  $H^0(S_{n+1}; M_L)$  over  $K_*$  is finite.

**Proof** We prove the lemma by induction on the dimension of  $M_L$ . Suppose that  $\dim M_L = 1$ . If  $H^0(S_{n+1}; M_L) = 0$ , then it is okay. Suppose that  $H^0(S_{n+1}; M_L) \neq 0$ . Take a nonzero  $a \in H^0(S_{n+1}; M_L)$ . Then  $M_L$  is isomorphic to  $L_*$  as a twisted  $L_*$ - $G_{n+1}$ -module. Hence this case follows from [Lemma 4.15](#).

Suppose that  $\dim M_L = n > 1$ , and that the lemma is true for  $M'_L$  of dimension  $< n$ . If  $H^0(S_{n+1}; M_L) = 0$ , then it is okay. Suppose that  $H^0(S_{n+1}; M_L) \neq 0$ . Let  $a \in H^0(S_{n+1}; M_L)$  be a nonzero element, and  $N_L$  the  $L_*$ -submodule generated by  $a$ . There is an exact sequence of  $K_*$ -modules:

$$0 \rightarrow H^0(S_{n+1}; N_L) \rightarrow H^0(S_{n+1}; M_L) \rightarrow H^0(S_{n+1}; M_L/N_L).$$

By hypothesis of the induction, the dimension of  $H^0(S_{n+1}; M_L/N_L)$  is finite and  $\dim H^0(S_{n+1}; N_L) = 1$ . Hence we obtain that  $\dim H^0(S_{n+1}; M_L)$  is finite. This completes the proof.  $\square$

**Remark 4.18** More precisely, we see that  $\dim_{K_*} H^0(S_{n+1}; N_L) \leq \dim_{L_*} N_L$  by the proof of [Lemma 4.17](#).

**Corollary 4.19** If  $M$  is a finitely generated discrete  $C_{E_*}$ -precomodule, then the dimension of  $\mathcal{F}(M)$  over  $K_*$  is finite.

The following lemma is fundamental on the topology of  $G_n$ .

**Lemma 4.20** (cf Hovey [9, Theorem A.2]) *A subgroup of  $G_n$  is open if and only if its index in  $G_n$  is finite.*

Let  $h \in G_n$  and  $\sigma$  the image of the projection  $G_n \rightarrow \text{Gal}$ . For  $g \in S_{n+1}$ ,  $g\sigma = \sigma g^\sigma$  in  $G_{n+1}$ , and  $gh = hg^\sigma$  in  $\mathcal{G}$ . Hence the following diagram commutes for all  $g \in S_{n+1}$ :

$$\begin{array}{ccc} L_* \widehat{\otimes}_{E_*} M & \xrightarrow{\sigma \otimes h} & L_* \widehat{\otimes}_{E_*} M \\ g \otimes g \downarrow & & \downarrow g^\sigma \otimes g^\sigma \\ L_* \widehat{\otimes}_{E_*} M & \xrightarrow{\sigma \otimes h} & L_* \widehat{\otimes}_{E_*} M. \end{array}$$

This diagram induces an action of  $G_n$  on  $\mathcal{F}(M)$ , and it is easy to check that  $\mathcal{F}(M)$  is a twisted  $K_*-G_n$ -module.

**Lemma 4.21** *If  $M$  is a finitely generated discrete  $C_{E_*}$ -precomodule, then  $\mathcal{F}(M)$  has a natural complete twisted  $K_*-G_n$ -module structure.*

**Proof** By Corollary 4.19,  $\mathcal{F}(M)$  is a twisted  $K_*-G_n$ -module of finite dimension. Then the action of  $G_n$  is continuous by Lemma 4.20.  $\square$

If  $M$  is a complete  $C_{E_*}$ -precomodule, then there is a fundamental system  $\{F^\lambda M\}$  of (open) neighborhoods at 0 consisting of  $E_*-G_{n+1}$ -submodules by Corollary 3.11. Hence there is an isomorphism

$$\mathcal{F}(M) \cong \varprojlim_{\lambda} \mathcal{F}(M/F^\lambda M).$$

We give  $\mathcal{F}(M)$  the inverse limit topology. Note that this topology is independent of a choice of fundamental system of neighborhood at 0. Furthermore, if  $M$  is profinite, then  $\mathcal{F}(M)$  is also profinite by Corollary 4.19, and complete twisted  $K_*-G_n$ -module by Lemma 4.21. Hence we obtain the following proposition.

**Proposition 4.22**  *$\mathcal{F}$  defines a symmetric monoidal functor from the category of profinite  $C_{E_*}$ -precomodules to the category of profinite  $C_{K_*}$ -comodules.*

**Proof** Since the construction of twisted  $K_*-G_n$ -module structure on  $\mathcal{F}(M)$  is natural, we see that  $\mathcal{F}$  defines a functor from the category of profinite  $C_{E_*}$ -precomodules to the category of profinite twisted  $K_*-G_n$ -modules, which is equivalent to the category of profinite  $C_{K_*}$ -comodules by Theorem 4.11. It is easy to check that the functor  $\mathcal{F}$  respects the monoidal structures.  $\square$

**Definition 4.23** Let  $\mathcal{C}_{\mathbb{E}}^f$  (resp.  $\mathcal{C}_{\mathbb{K}}^f$ ) be the category of finitely generated discrete  $C_{E_*}$ –precomodules (resp.  $C_{K_*}$ –(pre)comodules). Define  $\mathcal{C}_{\mathbb{E}}$  (resp.  $\mathcal{C}_{\mathbb{K}}$ ) to be the procategory of  $\mathcal{C}_{\mathbb{E}}^f$  (resp.  $\mathcal{C}_{\mathbb{K}}^f$ ), that is, the category of (small) cofiltered system of objects in  $\mathcal{C}_{\mathbb{E}}^f$  (resp.  $\mathcal{C}_{\mathbb{K}}^f$ ). These are symmetric monoidal categories.

For a finite  $Z$ ,  $E^*(Z)$  is a natural finitely generated discrete  $C_{E_*}$ –precomodule by [Proposition 3.14](#). Furthermore, if  $Z$  is a finite CW-complex, then  $E^*(Z)$  is a  $C_{E_*}$ –precomodule algebra.

**Definition 4.24** We define  $\mathbb{E}^*(X) \in \mathcal{C}_{\mathbb{E}}$  to be the system

$$\{E^*(Z)\}_{Z \in \Lambda(X)}$$

indexed by  $\Lambda(X)$ . We also define  $\mathbb{K}^*(X) \in \mathcal{C}_{\mathbb{K}}$  by the same manner.

Then we have

$$\varprojlim \mathbb{E}^*(X) \cong E^*(X),$$

$$\varprojlim \mathbb{K}^*(X) \cong K^*(X),$$

as profinite  $C_{E_*}$ –precomodules and profinite  $C_{K_*}$ –(pre)comodules, respectively. By [Corollary 4.19](#) and [Lemma 4.21](#), we can extend the functor  $\mathcal{F}$  from  $\mathcal{C}_{\mathbb{E}}$  to  $\mathcal{C}_{\mathbb{K}}$  by obvious way:

$$\mathcal{F}: \mathcal{C}_{\mathbb{E}} \longrightarrow \mathcal{C}_{\mathbb{K}}.$$

Note that  $\mathcal{F}$  is a monoidal functor.

By [\[22, Theorem 4.1\]](#), the generalized Chern character [\(5\)](#)

$$\Theta: E^*(X) \longrightarrow L^*(X)$$

induces a natural isomorphism of twisted  $L_*$ – $\mathcal{G}$ –modules:

$$L_* \otimes_{E_*} E^*(Z) \xrightarrow{\cong} L_* \otimes_{K_*} K^*(Z)$$

for finite  $Z$ . The following theorem is a reformulation of [\[22, Corollary 4.3\]](#).

**Theorem 4.25** For any spectrum  $X$ , the generalized Chern character  $\Theta$  induces a natural isomorphism in  $\mathcal{C}_{\mathbb{K}}$ :

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

If  $X$  is a space, then this is an isomorphism of cofiltered systems of finite  $C_{K_*}$ –comodule algebras.

## 5 Milnor operations

### 5.1 Complete co-operation ring

In this section we let  $A = E_{n+k}/I_n$  for some  $k \geq 0$ . Hence  $A = E$  if  $k = 1$  and  $A = K$  if  $k = 0$ . The coefficient ring  $A_* = \mathbf{F}[[u_n, \dots, u_{n+k-1}]] [u_A]$ ,  $|u_A| = -2$  is a graded complete Noetherian local ring with maximal ideal  $\mathfrak{m}_A = (u_n, \dots, u_{n+k-1})$ . Put  $G_A = G_{n+k}$  and  $C_{A_*} = C(G_A, A_*)$ . We denote by  $A \hat{\wedge} A$  the  $K(n+k)$ -localization of  $A \wedge A$ . Since  $A$  is a commutative ring spectrum, so is  $A \hat{\wedge} A$ . We define a graded commutative ring  $A_*^\vee(A)$  to be  $\pi_*(A \hat{\wedge} A)$ .

Since  $A$  is Landweber exact over  $P(n)$ , there is an isomorphism of commutative  $\mathbf{F}_p$ -algebras:

$$(6) \quad \pi_*(A \wedge A) \cong A_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} A_*.$$

**Lemma 5.1** *There is an isomorphism of graded commutative  $\mathbf{F}_p$ -algebras*

$$A_*^\vee(A) \cong A_*^c \hat{\otimes}_{P(n)_*} P(n)_*(P(n)) \hat{\otimes}_{P(n)_*} A_*^c,$$

where  $A_*^c$  is a graded topological ring  $A_*$  with  $\mathfrak{m}_A$ -adic topology.

**Proof** By [10, Proposition 7.10(e)], we see that  $A_*^\vee(A)$  is the  $I_{n+k}$ -adic completion of  $\pi_*(A \wedge A)$ . Hence the lemma follows from the isomorphism (6).  $\square$

By Lemma 5.1, we see that  $A_*^\vee(A)$  has a graded complete Hopf algebroid structure induced from  $P(n)_*(P(n))$ . We say that  $A_*^\vee(A)$  is the complete co-operation ring of  $A$ .

Let  $\Lambda_{\mathbf{Z}}$  be the graded commutative algebra over  $\mathbf{Z}$  generated by  $a_{(i)}$  for  $0 \leq i < n$ , where the degree of  $a_{(i)}$  is  $2p^i - 1$ . Hence  $\Lambda_{\mathbf{Z}}$  is an exterior algebra. For an evenly graded commutative ring  $R_*$ , we set  $\Lambda_{R_*} = R_* \otimes \Lambda_{\mathbf{Z}}$ . There is an isomorphism of commutative  $\mathbf{F}_p$ -algebras

$$P(n)_*(P(n)) \cong P(n)_*[t_1, t_2, \dots] \otimes \Lambda_{\mathbf{Z}},$$

where  $|t_i| = 2(p^i - 1)$ . Let  $C_{P(n)_*}$  be the  $P(n)_*$ -subalgebra of  $P(n)_*(P(n))$  generated by  $t_1, t_2, \dots$ :

$$C_{P(n)_*} = P(n)_*[t_1, t_2, \dots].$$

Then it is known that  $C_{P(n)_*}$  is a sub-Hopf algebroid of  $P(n)_*(P(n))$  (cf Wurgler [24]). Hence we can give  $A_*^c \hat{\otimes}_{P(n)_*} C_{P(n)_*} \hat{\otimes}_{P(n)_*} A_*^c$  the induced graded complete Hopf algebroid structure.



**Lemma 5.2** *There is an isomorphism of graded complete Hopf algebroids over  $\mathbf{F}_p$ :*

$$(A_*^c, C_{A_*}) \cong (A_*^c, A_*^c \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A_*^c).$$

**Proof** We let  $D_* = \pi_*(E_{n+k} \widehat{\wedge} E_{n+k})$ , where  $E_{n+k} \widehat{\wedge} E_{n+k}$  is the  $K(n+k)$ -localization of  $E_{n+k} \wedge E_{n+k}$ . Then  $D_* \cong E_{n+k,*}^c \widehat{\otimes}_{BP_*} BP_*(BP) \widehat{\otimes}_{BP_*} E_{n+k,*}^c$ . Hence  $(E_{n+k,*}^c, D_*)$  is a graded complete Hopf algebroid over  $\mathbf{Z}_{(p)}$ . Furthermore,

$$(E_{n+k,*}^c, D_*) \cong (E_{n+k,*}^c, C(G_{n+k}, E_{n+k,*}^c))$$

[4, 9]. Hence

$$D_*/I_n \cong A_*^c \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A_*^c$$

by Lemma 5.1. By Lemma 4.1,  $C(G_{n+k}, E_{n+k,*}) \cong C(G_{n+k}, \mathbf{Z}) \widehat{\otimes} E_{n+k,*}$ . This implies that  $C(G_{n+k}, E_{n+k,*})/I_n \cong C(G_{n+k}, A_*)$ . Hence we have the isomorphism  $C_{A_*} \cong A_*^c \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A_*^c$ . We can check that this isomorphism induces the desired isomorphism of Hopf algebroids.  $\square$

**Corollary 5.3** *There is an isomorphism of graded complete commutative  $\mathbf{F}_p$ -algebras:*

$$A_*^\vee(A) \cong C_{A_*} \widehat{\otimes} \Lambda_{\mathbf{Z}}.$$

Recall that  $A^*(X)^c = A_*^c \widehat{\otimes}_{A_*} A^*(X)$ . The natural  $P(n)_*(P(n))$ -comodule structure on  $P(n)^*(Z)$  gives a natural  $A_*^\vee(A)$ -comodule structure on  $A^*(Z)^c$ , for any finite spectrum  $Z$ . By Lemma 4.1 and Corollary 5.3, this induces an  $A_*^\vee(A)$ -comodule structure on  $A^*(X)^c$ :

$$\rho: A^*(X)^c \longrightarrow A_*^\vee(A) \widehat{\otimes}_{A_*^c} A^*(X)^c.$$

If  $X$  is a space, then  $\rho$  defines an  $A_*^\vee(A)$ -comodule algebra structure on  $A^*(X)^c$ .

In the following of this subsection we describe the comultiplication  $\psi$  on  $a_{(i)}^A$ . For  $0 \leq i < n$ , we set

$$b_{(i)}^A = u_A^{p^i} \otimes a_{(i)}.$$

Then  $|b_{(i)}^A| = -1$  for all  $i$ . In particular,  $u_A = u$  if  $A = E$  and  $u_A = w$  if  $A = K$ . We put

$$t_A(X) = \sum_{i \geq 0} \eta_{L^* F_A} t_i^A X^{p^i} \in C_{A_*} \llbracket X \rrbracket$$

$$\text{and} \quad b_A(X) = \sum_{i=0}^{n-1} b_{(i)}^A X^{p^i} \in A_*^\vee(A)[X],$$

where  $F_A$  is the base change of the universal deformation  $F_{n+k}$  on  $E_{n+k,0}$  to  $E_{n+k,0}/I_n$ .

The comodule algebra structure map  $\rho: A^*(\mathcal{X})^c \rightarrow A_*^\vee(A) \widehat{\otimes}_{A_*^c} A^*(\mathcal{X})^c$  on  $x_A$  is given by the following lemma (cf [2, Section 14]).

**Lemma 5.4**  $\psi_A(x_A) = t_A(x_A)$ .

The comodule algebra structure map  $\rho: A^*(\mathcal{Y})^c \rightarrow A_*^\vee(A) \widehat{\otimes}_{A_*^c} A^*(\mathcal{Y})^c$  on  $y_A$  is given by the following lemma (cf [2, Section 14]).

**Lemma 5.5**  $\rho(y_A) = 1 \otimes y_A + b_A(x_A)$ .

Let  $i_l$  and  $i_r$  be the left and right inclusion of  $A_*^\vee(A)$  into  $A_*^\vee(A) \widehat{\otimes}_{A_*^c} A_*^\vee(A)$ . The comultiplication map  $\psi$  on  $b_{(i)}^A$  is encoded in the following lemma.

**Lemma 5.6**  $\psi(b_A(X)) \equiv i_r(b_A)(X) + i_l(b_A)(i_r(t_A)(X)) \pmod{(X^{p^n})}$ .

**Proof** This follows from the fact that  $(\psi \otimes 1)\rho(y_A) = (1 \otimes \rho)\rho(y_A)$  and [Lemma 5.4](#) and [Lemma 5.5](#).  $\square$

**Lemma 5.7** Let  $F$  be a  $p$ -typical formal group law of strict height at least  $n$  over an  $\mathbb{F}_p$ -algebra  $R$ . Then for  $a_i \in R$  ( $0 \leq i < n$ ),

$$\sum_{i=0}^{n-1} a_i X^{p^i} \equiv a_0 X + a_1 X^p + \cdots + a_{n-1} X^{p^{n-1}} \pmod{(X^{p^n})}.$$

**Proof** This follows from the fact that  $F(X, Y) \equiv X + Y \pmod{(X, Y)^{p^n}}$ .  $\square$

The following theorem describes the structure of graded complete Hopf algebroid  $(A_*^c, A_*^\vee(A))$ .

**Theorem 5.8** The pair  $(A_*^c, A_*^\vee(A))$  is a graded complete Hopf algebroid over  $\mathbb{F}_p$ . There is an extension of graded complete Hopf algebroids

$$C_{A_*} \longrightarrow A_*^\vee(A) \longrightarrow \Lambda_{A_*^c},$$

where the algebra  $\Lambda_{A_*^c} = A_*^c \otimes \Lambda(b_{(0)}^A, \dots, b_{(n-1)}^A)$  is an exterior Hopf algebra over  $A_*^c$  generated by primitive elements  $b_{(i)}^A$  for  $0 \leq i < n$ . The comultiplication  $\psi$  and the counit  $\varepsilon$  on  $b_{(i)}^A$  for  $0 \leq i < n$  are given as follows:

$$\begin{aligned} \psi(b_{(i)}^A) &= 1 \otimes b_{(i)}^A + \sum_{j=0}^i b_{(j)}^A \otimes (t_{i-j}^A)^{p^j}, \\ \varepsilon(b_{(i)}^A) &= 0. \end{aligned}$$

**Proof** The comultiplication  $\psi$  on  $b_{(i)}^A$  is obtained by [Lemma 5.5](#) and [Lemma 5.7](#).  $\square$

## 5.2 Exterior algebras $\Lambda_{E_*}$ and $\Lambda_{K_*}$

We can give  $\Lambda_{A_*^c}$  a structure of right  $C_{A_*}$ -comodule algebra by

$$\rho_{C, \Lambda}^{\text{op}} : \Lambda_{A_*^c} \xrightarrow{i_{\Lambda}} A_*^{\vee}(A) \xrightarrow{\psi} A_*^{\vee}(A) \widehat{\otimes}_{A_*^c} A_*^{\vee}(A) \xrightarrow{\pi_{\Lambda} \otimes \pi_C} \Lambda_{A_*^c} \widehat{\otimes}_{A_*^c} C_{A_*},$$

where  $i_{\Lambda}$  is the canonical inclusion,  $\pi_C = 1_C \otimes \varepsilon_{\Lambda}$  and  $\pi_{\Lambda} = \varepsilon_C \otimes 1_{\Lambda}$ . Hence  $\Lambda_{A_*}$  is a profinite right  $C_{A_*}$ -precomodule algebra.

**Lemma 5.9**  $\rho_{\Lambda, C}^{\text{op}}(b_A(X)) \equiv b_A(t_A(X)) \pmod{(X^{p^n})}$ .

**Proof** This follows from [Lemma 5.6](#). □

The left  $A_*^c$ -module homomorphism  $\text{ev}(g) \circ \chi : C_{A_*} \rightarrow C_{A_*} \rightarrow A_*^c$  defines a right action of  $G_A$  on  $\Lambda_{A_*^c}$  by

$$\Lambda_{A_*^c} \xrightarrow{\rho_{\Lambda, C}^{\text{op}}} \Lambda_{A_*^c} \widehat{\otimes}_{A_*^c} C_{A_*} \xrightarrow{1 \otimes (\text{ev}(g) \circ \chi)} \Lambda_{A_*^c}.$$

Then  $\Lambda_{A_*^c}$  is a twisted  $A_*^c$ - $G_A$ -module.

**Corollary 5.10** For  $g \in G_A$ ,  $b_A^g(X) \equiv b_A(t_A(g)^{-1}(X)) \pmod{(X^{p^n})}$ .

**Proof** This follows from [Lemma 5.9](#) and  $t_A(g^{-1})^g(X) = t_A(g)^{-1}(X)$ . □

Since  $\Lambda_{E_*}$  is a twisted  $E_*$ - $G_{n+1}$ -module,  $\Lambda_{L_*} = L_* \otimes_{E_*} \Lambda_{E_*}$  is a twisted  $L_*$ - $\mathcal{G}$ -module. We define  $\widehat{b}(X) = \sum_{i=0}^{n-1} \widehat{b}_{(i)} X^{p^i} \in \Lambda_{L_*}[X]$  by

$$\widehat{b}(X) \equiv b_E(\Phi^{-1}(X)) \pmod{(X^{p^n})}.$$

**Lemma 5.11** For any  $g \in G_{n+1}$ ,  $\widehat{b}^g(X) = \widehat{b}(X)$ .

**Proof** By definition and [Corollary 5.10](#),  $\widehat{b}^g(X) \equiv b_E \circ t(g)^{-1} \circ (\Phi^{-1})^g(X) \pmod{(X^{p^n})}$ . By the diagram (3) in [Theorem 2.4](#),  $\Phi^g \circ t(g)(X) = \Phi(X)$ . This implies that  $t(g)^{-1} \circ (\Phi^g)^{-1}(X) = \Phi^{-1}(X)$ . Hence  $\widehat{b}^g(X) \equiv b \circ \Phi^{-1}(X) \pmod{(X^{p^n})}$ . □

By [Lemma 5.11](#), we see that the coefficients of  $\widehat{b}(X)$  are invariant under the action of  $G_{n+1}$ .

**Lemma 5.12**  $\mathcal{F}(\Lambda_{E_*}) = K_* \otimes \Lambda(\widehat{b}_{(0)}, \dots, \widehat{b}_{(n-1)})$  as a graded commutative ring.

**Proof** We have  $\widehat{b}_{(0)}, \dots, \widehat{b}_{n-1} \in \mathcal{F}(\Lambda_{E_*})$ . Since  $\widehat{b}_{(i)}$  is a linear combination of  $b_{(i)}^E, \dots, b_{(n-1)}^E$ , we see that  $K_* \otimes \Lambda(\widehat{b}_{(0)}, \dots, \widehat{b}_{(n-1)}) \subset \mathcal{F}(\Lambda_{E_*})$ . Then the lemma follows from the fact that  $\dim_{K_*} \mathcal{F}(\Lambda_{E_*}) \leq 2^n$  by [Remark 4.18](#).  $\square$

Recall that  $t_K(h)(X)$  is the automorphism  $t_K(h): H_n \longrightarrow H_n^h = H_n$  corresponding to  $h \in G_n$ .

**Lemma 5.13** For any  $h \in G_n$ , we have  $\widehat{b}^h(X) = \widehat{b} \circ t_K(h)^{-1}(X)$ .

**Proof** By definition and the fact that  $G_n$  acts on  $L$  as Galois group, we have  $\widehat{b}^h(X) \equiv b \circ (\Phi^{-1})^h(X) \pmod{(X^{p^n})}$ . By the diagram (4) in [Theorem 2.4](#),  $t_K(h) \circ \Phi(X) = \Phi^h(X)$ . This implies that  $(\Phi^h)^{-1}(X) = \Phi^{-1} \circ t_K(h)^{-1}(X)$ . Hence the congruence  $\widehat{b}^h(X) \equiv b \circ \Phi^{-1} \circ t_K(h)^{-1}(X) \pmod{(X^{p^n})}$  holds.  $\square$

**Theorem 5.14** As a  $C_{K_*}$ -comodule,  $\mathcal{F}(\Lambda_{E_*})$  is isomorphic to  $\Lambda_{K_*}$ .

**Proof** The map  $\widehat{b}_{(i)} \mapsto b_{(i)}^K$  gives an isomorphism of twisted  $K_*$ - $G_n$ -modules by [Corollary 5.10](#), [Lemma 5.12](#) and [Lemma 5.13](#).  $\square$

### 5.3 Milnor operations

Let  $A = E_{n+k}/I_n$  for some  $k \geq 0$ . In this section we study Milnor operations in  $A$ . We abbreviate  $C_{A_*}$  to  $C$  and  $\Lambda_{A_*^c}$  to  $\Lambda$ . In this section we discuss in the category of complete Hausdorff filtered  $A_*^c$ -modules. We recall that  $\Lambda$  is a Hopf algebra such that  $b_{(i)}^A$  is primitive for all  $i$ . We take monomials of  $b_{(i)}^A$  as a basis of  $\Lambda$ , and denote the dual of  $b_{(i)}^A$  by  $Q_i^A$  in the dual basis. Then the monomials of  $Q_i^A$  form the dual basis. We call  $Q_i^A$  the Milnor operations.

Let  $M$  be a left  $\Lambda$ -comodule with comodule structure map  $\rho$ . Then the Milnor operation  $Q_i^A$  defines a  $A_*^c$ -module homomorphism as follows:

$$M \xrightarrow{\rho} \Lambda \widehat{\otimes} M \xrightarrow{Q_i^A \otimes 1_M} M.$$

We abbreviate this homomorphism also to  $Q_i^A$ . Note that we write the action of  $Q_i^A$  from the right: if  $\rho(x) = 1 \otimes x + \sum_i a_{(i)}^A \otimes x_i + \dots$ , then  $(x)Q_i^A = (-1)^{|x|+1}x_i$ . There is a relation in the endomorphism ring of  $M$  for any  $i$  and  $j$ :

$$(7) \quad Q_i^A Q_j^A + Q_j^A Q_i^A = 0.$$

In particular,  $Q_i^A Q_i^A = 0$ . Conversely, if there are  $A_*^c$ -module homomorphisms  $Q_i^A$  for  $0 \leq i < n$  such that (7) holds, then we can construct a  $\Lambda$ -comodule structure on  $M$ , and this construction gives an equivalence of categories.

The category of complete  $\Lambda$ -comodules is symmetric monoidal under complete tensor product  $\widehat{\otimes}_{A_*^c}$  and unit object  $A_*^c$ .

**Lemma 5.15** *Let  $M$  and  $N$  be complete  $\Lambda$ -comodules. For any  $x \in M$  and  $y \in N$ ,  $(x \otimes y)Q_i^A = x \otimes (y)Q_i^A + (-1)^{|y|}(x)Q_i^A \otimes y$  in  $M \widehat{\otimes} N$ .*

**Proof** Let  $\rho_M(x) = 1 \otimes x + \sum_i a_{(i)}^A \otimes x_i + \cdots$  with  $x_i = (-1)^{|x|+1}(x)Q_i^A$ , and let  $\rho_N(y) = 1 \otimes y + \sum_i a_{(i)}^A \otimes y_i + \cdots$  with  $y_i = (-1)^{|y|+1}(y)Q_i^A$ . Then

$$\rho_{M \widehat{\otimes} N}(x \otimes y) = 1 \otimes x \otimes y + (-1)^{|x|} \sum_i a_{(i)}^A \otimes x \otimes y_i + \sum_i a_{(i)}^A \otimes x_i \otimes y + \cdots.$$

Hence we have  $(x \otimes y)Q_i^A = (-1)^{|x|+|y|+1}((-1)^{|x|}x \otimes y_i + x_i \otimes y)$ , which equals  $x \otimes (y)Q_i^A + (-1)^{|y|}(x)Q_i^A \otimes y$ .  $\square$

We say that a natural endomorphism  $Q$  of complete  $A_*^c$ -modules is a derivation of odd degree with respect to exterior products if  $(x \otimes y)Q = x \otimes (y)Q + (-1)^{|y|}(x)Q \otimes y$  for any  $x \in M$  and  $y \in N$ . Hence the Milnor operations  $Q_i^A$  is a derivation of odd degree with respect to exterior products.

Let  $\Lambda^*$  be the dual module of  $\Lambda$ :  $\Lambda^* = \text{Hom}_{A_*^c}(\Lambda, A_*^c)$ . Then  $\Lambda^*$  is also a Hopf algebra over  $A_*^c$ , and  $\Lambda^* \cong A_*^c \otimes \Lambda(Q_0^A, \dots, Q_{n-1}^A)$  such that  $Q_i^A$  are primitive for all  $i$ . Recall that  $\Lambda$  is a twisted  $A_*^c$ - $G_A$ -module. We can also define a twisted  $A_*^c$ - $G_A$ -module structure on  $\Lambda^*$  by

$$(\lambda)(\theta^g) = ((\lambda \cdot g^{-1})\theta)g,$$

for  $\theta \in \Lambda^*$ ,  $g \in G_A$ ,  $\lambda \in \Lambda$ .

**Lemma 5.16** *For  $g \in G_A$ ,*

$$(Q_i^A)^g = \sum_{j=i}^{n-1} t_{j-i}^A(g)^{p^j} Q_j^A.$$

**Proof** This follows from [Corollary 5.10](#).  $\square$

Let  $M$  be a profinite  $A_*^c(A)$ -comodule. Then  $M$  is a twisted  $A_*^c$ - $G_A$ -module and  $\Lambda$ -comodule. The following proposition gives us an interaction of the actions of  $G_A$  and  $Q_i^A$  on  $M$ .

**Lemma 5.17** *Let  $M$  be a profinite  $A_*^\vee(A)$ -comodule. For  $x \in M$  and  $g \in G_A$ ,*

$$((x)Q_i^A)g = ((x)g)(Q_i^A)^g.$$

**Proof** By Lemma 5.16, we see that the map

$$\theta_1 : A_*^\vee(A) \xrightarrow{\psi} A_*^\vee(A) \widehat{\otimes} A_*^\vee(A) \xrightarrow{(Q_i^A \circ \pi_\Lambda) \otimes 1} A_*^\vee(A) \xrightarrow{\text{ev}(g) \circ \pi_C} A_*^c$$

is equal to the map

$$\theta_2 : A_*^\vee(A) \xrightarrow{\psi} A_*^\vee(A) \widehat{\otimes} A_*^\vee(A) \xrightarrow{(\text{ev}(g) \circ \pi_C) \otimes 1} A_*^\vee(A) \xrightarrow{(Q_i^A)^g \circ \pi_\Lambda} A_*^c.$$

Hence  $((x)Q_i^A)g = (\theta_1 \circ \rho)(x) = (\theta_2 \circ \rho)(x) = ((x)g)(Q_i^A)^g$ .  $\square$

These are all relations on the  $A_*^\vee(A)$ -comodule  $M$  between the  $G_A$ -action and the  $\Lambda^*$ -action. We give interpretation of these relations in terms of comodule structures in Section 5.4.

In the following lemma we show that a derivation of odd degree with respect to exterior products in the category of stable cohomology operations of  $K^*(-)$  is characterized by the action on  $y_K \in K^*(\mathcal{Y})$ .

**Lemma 5.18** *Let  $Q$  be an odd degree stable cohomology operation of  $K^*(-)$ . Suppose that  $Q$  is a derivation with respect to exterior product. Then  $Q$  is characterized by the action on  $y_K \in K^1(\mathcal{Y})$ .*

**Proof** A stable cohomology operation  $Q \in K^*(K)$  is a derivation if and only if  $Q$  is primitive in  $K^*(K)$ . Since  $K_*(K)$  is free over  $K_*$ , the primitive submodule  $P(K^*(K))$  is the dual of the indecomposable quotient  $Q(K_*(K))$  of the co-operation ring  $K_*(K)$ . Recall the isomorphism  $K_*(K) \cong C_{K_*} \otimes_{K_*} \Lambda_{K_*}$ . Then we have  $Q(K_*(K)) \cong Q(C_{K_*}) \oplus Q(\Lambda_{K_*}) \cong Q(\Lambda_{K_*})$ , and  $Q(\Lambda_{K_*})$  is isomorphic to  $K_*\{a_{(0)}^K, \dots, a_{(n-1)}^K\}$ . Hence  $Q$  is a linear combination  $\sum_{i=0}^{n-1} q_i Q_i^K$  with  $q_i \in K_*$ . Since we know that  $Q_i^K(y_K) = x_K^{p^i}$ , we have  $Q(y_K) = \sum_{i=0}^{n-1} q_i x_K^{p^i}$  in  $K^*(\mathcal{Y})$ . Since  $x_K^{p^i}$  for  $0 \leq i < n-1$  are linearly independent, this uniquely determines  $q_i$ . Hence  $Q$  is characterized by the action of  $y_K$ .  $\square$

## 5.4 Complete $A_*^\vee(A)$ -comodules

Let  $A = E_{n+k}/I_n$  for some  $k \geq 0$ . In this section we give a description of complete  $A_*^\vee(A)$ -comodules in terms of  $C_{A_*}$ -comodule structure and  $\Lambda_{A_*^c}$ -comodule structure.

In this section we discuss in the category of complete Hausdorff filtered  $A_*^c$ -modules, and abbreviate  $C_{A_*}$  to  $C$  and  $\Lambda_{A_*^c}$  to  $\Lambda$ .

Let  $M$  be a complete  $A_*^\vee(A)$ -comodule with  $\rho_M: M \rightarrow A_*^\vee(A) \hat{\otimes} M$ . By [Theorem 5.8](#),  $A_*^\vee(A) \cong C \hat{\otimes} \Lambda$  as an  $\mathbb{F}_p$ -algebra, and there is an extension of complete Hopf algebroids:

$$(8) \quad C \longrightarrow A_*^\vee(A) \xrightarrow{\pi_\Lambda} \Lambda.$$

Hence  $M$  is a  $\Lambda$ -comodule by

$$\rho_{\Lambda, M} \xrightarrow{\rho_M} A_*^\vee(A) \hat{\otimes} M \xrightarrow{\pi_\Lambda} \Lambda \hat{\otimes} M.$$

The counit of  $\Lambda$  induces a morphism of Hopf algebroid  $\pi_C: A_*^\vee(A) \rightarrow C$ , which is a splitting of the above extension (8). Then  $M$  is also a  $C$ -comodule by

$$\rho_{C, M}: M \xrightarrow{\rho_M} A_*^\vee(A) \hat{\otimes} M \xrightarrow{\pi_C} C \hat{\otimes} M.$$

We recall that  $\Lambda$  is a (left)  $C$ -comodule algebra by the structure map

$$\rho_{C, \Lambda}: \Lambda \xrightarrow{i_\Lambda} C \hat{\otimes} \Lambda \xrightarrow{\psi} (C \hat{\otimes} \Lambda) \hat{\otimes} (C \hat{\otimes} \Lambda) \xrightarrow{\pi_\Lambda \otimes \pi_C} \Lambda \hat{\otimes} C \xrightarrow{\tau} C \hat{\otimes} \Lambda,$$

where  $i_\Lambda$  is the canonical inclusion and  $\tau$  is given by  $\lambda \otimes c \mapsto \chi(c) \otimes \lambda$ . For a complete  $C$ -comodule  $M$ , we denote by  $\rho_{C, \Lambda \hat{\otimes} M}$  the  $C$ -comodule structure map of the tensor product of  $\Lambda$  and  $M$ .

**Lemma 5.19** *Let  $M$  be a complete  $A_*^\vee(A)$ -comodule. Then  $\rho_{\Lambda, M}$  is a morphism of  $C$ -comodules. In other words, the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\rho_{\Lambda, M}} & \Lambda \hat{\otimes} M \\ \rho_{C, M} \downarrow & & \downarrow \rho_{C, \Lambda \hat{\otimes} M} \\ C \hat{\otimes} M & \xrightarrow{1_C \otimes \rho_{\Lambda, M}} & C \hat{\otimes} \Lambda \hat{\otimes} M. \end{array}$$

Furthermore,  $\rho_{C, \Lambda \hat{\otimes} M} \circ \rho_{\Lambda, M} = (1_C \otimes \rho_{\Lambda, M}) \circ \rho_{C, M}$  is the  $A_*^\vee(A)$ -comodule structure map  $\rho_M$ .

**Proof** Let  $f = (\pi_\Lambda \otimes \pi_C) \circ \psi: A_*^\vee(A) \rightarrow \Lambda \hat{\otimes} C$ . By the co-associativity of  $A_*^\vee(A)$ -comodule  $M$ , the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho_{\Lambda, M}} & \Lambda \hat{\otimes} M \\ \rho_M \downarrow & & \downarrow 1_\Lambda \otimes \rho_{C, M} \\ C \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{f \otimes 1_M} & \Lambda \hat{\otimes} C \hat{\otimes} M. \end{array}$$

Let  $g = (1_C \otimes 1_\Lambda \otimes \varepsilon_C) \circ \rho_{C, \Lambda \hat{\otimes} C} : \Lambda \hat{\otimes} C \rightarrow C \hat{\otimes} \Lambda$ . Then we can check that  $g \circ f$  is the identity map of  $C \hat{\otimes} \Lambda$ . Since  $(g \otimes 1_M) \circ (1_\Lambda \otimes \rho_{C, M}) = \rho_{C, \Lambda \hat{\otimes} M}$ , we obtain that  $\rho_M = \rho_{C, \Lambda \hat{\otimes} M} \circ \rho_{\Lambda, M}$ .

Let  $h = (\pi_C \otimes \pi_\Lambda) \circ \psi : A_*^\vee(A) \rightarrow A_*^\vee(A)$ . By the coassociativity of  $A_*^\vee(A)$ -comodule  $M$ , the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho_{C, M}} & C \hat{\otimes} M \\ \rho_M \downarrow & & \downarrow 1_C \otimes \rho_{\Lambda, M} \\ C \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{h \otimes 1_M} & C \hat{\otimes} \Lambda \hat{\otimes} M. \end{array}$$

But it is easy to check that  $h$  is the identity map of  $A_*^\vee(A)$ . Hence we obtain that  $\rho_M = (1_C \otimes \rho_{\Lambda, M}) \circ \rho_{C, M}$ . This completes the proof.  $\square$

**Definition 5.20** We say that a complete module  $M$  is a  $C$ - $\Lambda$ -comodule if  $M$  is a  $C$ -comodule and also a  $\Lambda$ -comodule such that the structure map of  $\Lambda$ -comodule  $\rho_{\Lambda, M}$  is a map of  $C$ -modules.

**Corollary 5.21** A complete  $A_*^\vee(A)$ -comodule has a natural  $C$ - $\Lambda$ -comodule structure.

Let  $\rho_{\Lambda, C \hat{\otimes} \Lambda} = (\pi_\Lambda \otimes 1 \otimes 1) \circ \psi : C \hat{\otimes} \Lambda \rightarrow \Lambda \hat{\otimes} C \hat{\otimes} \Lambda$ .

**Lemma 5.22** The following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\psi_\Lambda} & \Lambda \hat{\otimes} \Lambda \\ \rho_{C, \Lambda} \downarrow & & \downarrow 1_\Lambda \otimes \rho_{C, \Lambda} \\ C \hat{\otimes} \Lambda & \xrightarrow{\rho_{\Lambda, C \hat{\otimes} \Lambda}} & \Lambda \hat{\otimes} C \hat{\otimes} \Lambda. \end{array}$$

**Proof** Note that this is a diagram of  $\mathbf{F}_p$ -algebras. So it is sufficient to show the equality  $f(b_{(i)}) = g(b_{(i)})$  holds for all  $0 \leq i < n$ , where  $f = (1_\Lambda \otimes \rho_{C, \Lambda}) \circ \psi_\Lambda$  and  $g = \rho_{\Lambda, C \hat{\otimes} \Lambda} \circ \rho_{C, \Lambda}$ . We easily obtain that

$$f(b_{(i)}) = b_{(i)} \otimes 1 \otimes 1 + 1 \otimes \sum_{j=0}^i \chi(s_{i-j}) p^j \otimes b_{(j)}.$$



On the other hand,

$$\begin{aligned}
 g(b_{(i)}) &= \sum_{j=0}^i (1 \otimes \chi(s_{i-j})^{p^j} \otimes 1) \cdot (1 \otimes 1 \otimes b_{(j)} + \sum_{k=0}^j b_{(k)} \otimes s_{j-k}^{p^k} \otimes 1) \\
 &= \sum_{j=0}^i 1 \otimes \chi(s_{i-j})^{p^j} \otimes b_{(j)} + \sum_{k=0}^i \sum_{j=k}^i b_{(k)} \otimes \left( s_{j-k} \chi(s_{i-j})^{p^{j-k}} \right)^{p^k} \otimes 1 \\
 &= \sum_{j=0}^i 1 \otimes \chi(s_{i-j})^{p^j} \otimes b_{(j)} + b_{(i)} \otimes 1 \otimes 1.
 \end{aligned}$$

This completes the proof.  $\square$

**Corollary 5.23** *If  $M$  is a complete  $C$ -comodule, then the following diagram commutes:*

$$(9) \quad \begin{array}{ccc} \Lambda \hat{\otimes} M & \xrightarrow{\psi_\Lambda \otimes 1_M} & \Lambda \hat{\otimes} \Lambda \hat{\otimes} M \\ \rho_{C, \Lambda \hat{\otimes} M} \downarrow & & \downarrow 1_\Lambda \otimes \rho_{C, \Lambda \hat{\otimes} M} \\ C \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{\rho_{\Lambda, C \hat{\otimes} \Lambda} \otimes 1_M} & \Lambda \hat{\otimes} C \hat{\otimes} \Lambda \hat{\otimes} M. \end{array}$$

**Lemma 5.24** *Let  $M$  be a complete  $C$ - $\Lambda$ -comodule with  $C$ -comodule structure map  $\rho_{C, M}: M \rightarrow C \hat{\otimes} M$ . Then the following diagram commutes:*

$$\begin{array}{ccccc} M & \xrightarrow{\rho_{\Lambda, M}} & \Lambda \hat{\otimes} M & \xrightarrow{\rho_{C, \Lambda \hat{\otimes} M}} & C \hat{\otimes} \Lambda \hat{\otimes} M \\ \downarrow \rho_{\Lambda, M} & & \downarrow 1_\Lambda \otimes \rho_{\Lambda, M} & & \downarrow 1_\Lambda \otimes \rho_{\Lambda, M} \\ \Lambda \hat{\otimes} M & \xrightarrow{\psi_\Lambda \otimes 1} & \Lambda \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{\rho_{C, \Lambda \hat{\otimes} \Lambda \hat{\otimes} M}} & C \hat{\otimes} \Lambda \hat{\otimes} \Lambda \hat{\otimes} M \\ \downarrow \rho_{C, \Lambda \hat{\otimes} M} & & \downarrow 1_\Lambda \otimes \rho_{C, \Lambda \hat{\otimes} M} & & \downarrow 1_\Lambda \otimes \rho_{C, \Lambda \hat{\otimes} M} \\ C \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{\rho_{\Lambda, C \hat{\otimes} \Lambda} \otimes 1} & \Lambda \hat{\otimes} C \hat{\otimes} \Lambda \hat{\otimes} M & \xrightarrow{\rho_{C, \Lambda \hat{\otimes} C} \otimes 1 \otimes 1} & C \hat{\otimes} \Lambda \hat{\otimes} C \hat{\otimes} \Lambda \hat{\otimes} M \end{array}$$

**Proof** The top left square commutes since  $M$  is a  $\Lambda$ -comodule. From the assumption that  $\rho_{\Lambda, M}$  is a morphism of  $C$ -comodules, so is  $1_\Lambda \otimes \rho_{\Lambda, M}$ . Hence we see that the top right square commutes. The bottom left square commutes by Corollary 5.23. Since  $\rho_{C, \Lambda \hat{\otimes} M}$  is a morphism of  $C$ -comodules, so is  $1_\Lambda \otimes \rho_{C, \Lambda \hat{\otimes} M}$ . Hence the bottom right square commutes. This completes the proof.  $\square$

**Lemma 5.25** *The map  $(\rho_{C, \Lambda \hat{\otimes} C} \otimes 1_\Lambda) \circ (\rho_{\Lambda, C \hat{\otimes} \Lambda})$  is the comultiplication  $\psi$ .*

**Proof** Let  $f = (\rho_{C, \Lambda \widehat{\otimes} C} \otimes 1_\Lambda) \circ (\rho_{\Lambda, C \widehat{\otimes} \Lambda})$ . Since  $f$  is a map of  $\mathbb{F}_p$ -algebras, it is sufficient to show that  $f(c) = \psi(c)$  for all  $c \in C$  and  $f(b_{(i)}) = \psi(b_{(i)})$  for all  $0 \leq i < n$ . It is easy to check that  $f(c) = \psi(c)$ . On the other hand,

$$\begin{aligned} f(b_{(i)}) &= \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^{i-j} \chi(s_{j-k})^{p^k} s_{i-j-l}^{p^j} \otimes b_{(k)} \otimes s_l^{p^{i-l}} \otimes 1 + 1 \otimes 1 \otimes 1 \otimes b_{(i)} \\ &= \sum_{\substack{k, l \geq 0 \\ k+l \leq i}} \left( \sum_{j=k}^{i-l} \chi(s_{j-k})^{p^k} s_{i-j-l}^{p^j} \right)^{p^k} \otimes b_{(k)} \otimes s_l^{p^{i-l}} + 1 \otimes 1 \otimes 1 \otimes b_{(i)} \\ &= 1 \otimes 1 \otimes 1 \otimes b_{(i)} + \sum_{k=0}^i 1 \otimes b_{(k)} \otimes s_{i-k}^{p^k} \otimes 1. \end{aligned}$$

Hence  $f(b_{(i)}) = \psi(b_{(i)})$ . This completes the proof.  $\square$

Let  $M$  be a complete  $C$ - $\Lambda$ -comodule with  $C$ -comodule structure map  $\rho_{C, M}: M \rightarrow C \widehat{\otimes} M$ . We define a map  $\rho_M: M \rightarrow C \widehat{\otimes} \Lambda \widehat{\otimes} M$  by

$$\rho_M: M \xrightarrow{\rho_{\Lambda, M}} \Lambda \widehat{\otimes} M \xrightarrow{\rho_{C, \Lambda \widehat{\otimes} M}} C \widehat{\otimes} \Lambda \widehat{\otimes} M.$$

By [Lemma 5.24](#) and [Lemma 5.25](#), we see that  $\rho_M$  gives  $M$  a complete  $A_*^\vee(A)$ -comodule structure.

**Proposition 5.26** *Let  $M$  be a complete  $C$ - $\Lambda$ -comodule. Then  $M$  has a natural  $A_*^\vee(A)$ -comodule structure  $\rho_M$  such that the induced  $C$ - $\Lambda$ -comodule structure coincides with the given one.*

Note that if  $M$  is a complete  $C$ - $\Lambda$ -comodule obtained from a complete  $A_*^\vee(A)$ -comodule, then the induced  $A_*^\vee(A)$ -comodule structure coincides with the given one by [Lemma 5.19](#).

By summarizing the results in this section, we obtain the following theorem.

**Theorem 5.27** *There is an equivalence of symmetric monoidal categories between the category of complete  $A_*^\vee(A)$ -comodules and the category of complete  $C$ - $\Lambda$ -comodules.*

## 6 Main theorem

### 6.1 Symmetric monoidal functor $\mathcal{F}$

In [Proposition 4.22](#) we showed that  $\mathcal{F}$  is a monoidal functor from the category of profinite  $C_{E_*}$ -precomodules to the category of profinite  $C_{K_*}$ -comodules. In this section

we show that the functor  $\mathcal{F}$  extends to a monoidal functor from the category of profinite  $E_*^\vee(E)$ -precomodules to the category of profinite  $K_*(K)$ -comodules.

We let  $M$  be a profinite  $E_*^\vee(E)$ -precomodule. Then  $M$  is a profinite  $C_{E_*}$ -precomodule and also a  $\Lambda_{E_*}$ -comodule such that the  $\Lambda_{E_*}$ -comodule structure map  $\rho_M: M \rightarrow \Lambda_{E_*} \widehat{\otimes} M$  is a map of profinite  $C_{E_*}$ -precomodules by [Corollary 5.21](#). We note that  $\Lambda_{E_*}$  is a  $C_{E_*}$ -precomodule and there is an isomorphism of  $C_{K_*}$ -comodules:  $\mathcal{F}(\Lambda_{E_*}) \cong \Lambda_{K_*}$  by [Theorem 5.14](#).

**Lemma 6.1** *If  $M$  is a profinite  $C_{E_*}$ -precomodule, then the natural map*

$$\Lambda_{K_*} \widehat{\otimes}_{K_*} \mathcal{F}(M) \xrightarrow{\cong} \mathcal{F}(\Lambda_{E_*}) \widehat{\otimes}_{K_*} \mathcal{F}(M) \longrightarrow \mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} M)$$

*is an isomorphism of  $C_{K_*}$ -comodules.*

**Proof** Since  $\mathcal{F}(\Lambda_{E_*})$  is isomorphic to  $\Lambda_{K_*}$  as a  $C_{K_*}$ -comodule,  $\Lambda_{E_*} \widehat{\otimes}_{E_*} L_* \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} L_*$  as twisted  $L_*$ - $\mathcal{G}$ -modules. Hence there are isomorphisms of twisted  $L_*$ - $\mathcal{G}$ -modules:

$$\begin{aligned} \Lambda_{E_*} \widehat{\otimes}_{E_*} M \widehat{\otimes}_{E_*} L_* &\cong (\Lambda_{E_*} \widehat{\otimes}_{E_*} L_*) \widehat{\otimes}_{L_*} (M \widehat{\otimes}_{E_*} L_*) \\ &\cong (\Lambda_{K_*} \widehat{\otimes}_{K_*} L_*) \widehat{\otimes}_{L_*} (M \widehat{\otimes}_{E_*} L_*) \\ &\cong \Lambda_{K_*} \widehat{\otimes}_{K_*} M \widehat{\otimes}_{E_*} L_*. \end{aligned}$$

By taking  $S_{n+1}$ -invariant submodules, we obtain an isomorphism of twisted  $K_*$ - $G_n$ -modules:  $\mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} M) \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} \mathcal{F}(M)$ . This implies that the above map is an isomorphism of profinite  $C_{K_*}$ -comodules.  $\square$

By [Lemma 6.1](#), we obtain a map

$$\mathcal{F}(\psi_{\Lambda_{E_*}}): \Lambda_{K_*} \cong \mathcal{F}(\Lambda_{E_*}) \longrightarrow \mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} \Lambda_{E_*}) \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} \Lambda_{K_*}.$$

**Lemma 6.2** *The map  $\mathcal{F}(\psi_{\Lambda_{E_*}})$  coincides with the comultiplication map  $\psi_{\Lambda_{K_*}}$  on  $\Lambda_{K_*}$ .*

**Proof** This follows from the fact that the algebra generators  $\widehat{b}_{(i)}^E$  of  $\mathcal{F}(\Lambda_{E_*})$  are given by linear combinations of the algebra generators  $b_{(i)}^E$  of  $\Lambda_{E_*}$  with coefficients in  $L_*$  (see [Lemma 5.12](#)).  $\square$

**Corollary 6.3** *Let  $M$  be a profinite  $E_*^\vee(E)$ -precomodule with corresponding  $\Lambda_{E_*}$ -comodule structure map  $\rho_M: M \rightarrow \Lambda_{E_*} \widehat{\otimes}_{E_*} M$ . Then the map  $\mathcal{F}(\rho_M): \mathcal{F}(M) \rightarrow \mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} M) \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} \mathcal{F}(M)$  defines a natural  $\Lambda_{K_*}$ -comodule structure on  $\mathcal{F}(M)$ .*

**Proposition 6.4** *If  $M$  is a profinite  $E_*^\vee(E)$ -precomodule, then  $\mathcal{F}(M)$  has a natural  $K_*(K)$ -comodule structure.*

**Proof** Since  $\mathcal{F}$  is a functor and the  $\Lambda_{E_*}$ -comodule structure map  $\rho_M: M \rightarrow \Lambda_E \hat{\otimes} M$  is a map of  $C_{E_*}$ -precomodule,  $\mathcal{F}(\rho_M)$  is a map of  $C_{K_*}$ -comodules. Hence the proposition follows from [Theorem 5.27](#).  $\square$

**Corollary 6.5**  *$\mathcal{F}$  extends to a symmetric monoidal functor from the category of profinite  $E_*^\vee(E)$ -precomodules to the category of profinite  $K_*(K)$ -comodules.*

**Proof** By [Proposition 6.4](#), we see that  $\mathcal{F}$  extends to a functor from the category of profinite  $E_*^\vee(E)$ -precomodules to the category of profinite  $K_*(K)$ -comodules. It is easy to check that  $\mathcal{F}$  respects the symmetric monoidal structures.  $\square$

## 6.2 Main theorem

In this section we prove the main theorem ([Theorem 6.11](#)). The theorem states that for any spectrum  $X$ ,  $\mathcal{F}(\mathbb{E}^*(X))$  is naturally isomorphic to  $\mathbb{K}^*(X)$  as a cofiltered system of finitely generated discrete  $K_*(K)$ -comodules. Furthermore, if  $X$  is a space, then this equivalence respects the graded commutative ring structures.

**Definition 6.6** Let  $\mathcal{M}_{\mathbb{E}}^f$  (resp.  $\mathcal{M}_{\mathbb{K}}^f$ ) be the category of finitely generated discrete  $E_*^\vee(E)$ -precomodules (resp.  $K_*(K)$ -(pre)comodules). We define  $\mathcal{M}_{\mathbb{E}}$  (resp.  $\mathcal{M}_{\mathbb{K}}$ ) to be the procategory of  $\mathcal{M}_{\mathbb{E}}^f$  (resp.  $\mathcal{M}_{\mathbb{K}}^f$ ).

By [Corollary 6.5](#), we can extend the functor  $\mathcal{F}$  from  $\mathcal{M}_{\mathbb{E}}$  to  $\mathcal{M}_{\mathbb{K}}$  by obvious way:

$$\mathcal{F}: \mathcal{M}_{\mathbb{E}} \longrightarrow \mathcal{M}_{\mathbb{K}}.$$

Note that  $\mathcal{F}$  is a monoidal functor. As in the cases of  $\mathcal{C}_{\mathbb{E}}$  and  $\mathcal{C}_{\mathbb{K}}$ , we have the following lemma.

**Lemma 6.7** *For any spectrum  $X$ ,  $\mathbb{E}^*(X) \in \mathcal{M}_{\mathbb{E}}$  and  $\mathbb{K}^*(X) \in \mathcal{M}_{\mathbb{K}}$ .*

Hence  $\mathcal{F}(\mathbb{E}^*(X)) \in \mathcal{M}_{\mathbb{K}}$ . The natural  $\Lambda_{K_*}$ -comodule structure on  $\mathcal{F}(\mathbb{E}^*(X))$  gives natural  $K_*$ -module homomorphisms  $\hat{Q}_i$  on  $K^*(X) = \varprojlim \mathcal{F}(\mathbb{E}^*(X))$  for  $0 \leq i < n$  with respect to the algebra generators  $b_{(i)}^K$  of  $\Lambda_{K_*}$ .

**Lemma 6.8** *For  $0 \leq i < n$ ,  $\hat{Q}_i$  is a stable cohomology operation on  $K^*(X)$ .*

**Proof** It is sufficient to show that  $\widehat{Q}_i$  commutes with the suspension isomorphism  $\Sigma$ . Let  $s \in \widetilde{K}^1(S^1)$  the canonical generator  $\Sigma(1)$ . Then the suspension isomorphism is given by the (exterior) product with  $s$ . Since  $\widehat{Q}_i$  is an odd degree operation,  $\widehat{Q}_i$  acts on  $s$  trivially. Hence we see that  $\widehat{Q}_i$  commutes with the product with  $s$  since  $\widehat{Q}_i$  is a derivation. This completes the proof.  $\square$

Recall that  $\mathcal{Y}$  is the lens space  $S^{2p^n-1}/C_p$ , and  $K^*(\mathcal{Y}) = \Lambda(u_k) \otimes K_*[x_K]/(x_K^{p^n})$ .

**Lemma 6.9** For  $0 \leq i < n$ ,  $\widehat{Q}_i(u_K) = x_K^{p^i}$ .

**Proof** By Lemma 5.5,  $\rho(y_E) = 1 \otimes y_E + b^E(x_E)$ . Since  $b_E(X) = \widehat{b}(\Phi(X)) \bmod (X^{p^n})$  by definition, we have  $\rho(1 \otimes u_E) = 1 \otimes 1 \otimes u_E + \widehat{b}^E(\Phi(x_E))$ . From that fact that  $u_K = 1 \otimes u_E$  and  $\Phi(x_E) = x_K$ , we obtain that  $\widehat{Q}_i(u_K) = x_K^{p^i}$ .  $\square$

**Corollary 6.10** For  $0 \leq i < n$ ,  $\widehat{Q}_i = Q_i^K$ .

**Proof** By Lemma 6.8,  $\widehat{Q}_i$  is an odd degree stable cohomology operation, which is a derivation with respect to the (exterior) product. Hence  $\widehat{Q}_i$  is characterized by the action on  $u_K \in K^1(\mathcal{Y})$ . Then the corollary follows from Lemma 6.9.  $\square$

Recall that the generalized Chern character (5)

$$\Theta: E^*(X) \longrightarrow L^*(X)$$

induces a natural isomorphism in  $\mathcal{C}_{\mathbb{K}}$

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X)$$

by Theorem 4.25. The following is our main theorem of this note.

**Theorem 6.11** The generalized Chern character  $\Theta$  induces a natural isomorphism in  $\mathcal{M}_{\mathbb{K}}$ :

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

If  $X$  is a space, then this is an isomorphism of cofiltered systems of finite  $K_*(K)$ -comodule algebras.

**Proof** By Theorem 4.25, there is a natural isomorphism  $\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X)$  in  $\mathcal{C}_{\mathbb{K}}$ . Corollary 6.10 implies that the isomorphism  $\mathcal{F}(E^*(Z)) \cong K^*(Z)$  respects the  $\Lambda_{K_*}$ -comodule structures for all  $Z \in \Lambda(X)$ . Hence the theorem follows from Theorem 5.27.  $\square$

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